THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: JAMES A. BROOKE

TITLE OF THESIS: CLIFFORD ALGEBRAS, SPIN GROUPS AND GALILEI INVARIANCE - NEW PERSPECTIVES

DEGREE FOR WHICH THESIS PRESENTED: DOCTOR OF PHILOSOPHY

YEAR THIS DEGREE GRANTED: 1980

THE UNIVERSITY OF ALBERTA

CLIFFORD ALGEBRAS, SPIN GROUPS AND GALILEI INVARIANCE - NEW PERSPECTIVES

BY

(C) JAMES A. BROOKE

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1980

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled CLIFFORD ALGEBRAS, SPIN GROUPS AND GALILEI INVARIANCE - NEW PERSPECTIVES submitted by JAMES A. BROOKE in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The title of this thesis suggests a three-way interplay of ideas whose two-way components will be described.

Clifford Algebras and Spin Groups: A spin group is, at least from one point of view, roughly a two-fold covering group of an orthogonal group; and the most familiar examples of orthogonal groups are the groups of rotations in the Euclidean plane and space. A Clifford algebra is a finite dimensional algebra whose generators satisfy certain quadratic relations which derive from orthogonality statements in an associated orthogonal space. When the orthogonal space is non-degenerate, there is a nice construction of the two-fold covering of its orthogonal group which employs the Clifford algebra. Curiously, the relationship in the degenerate situation seems not to have been investigated in a systematic way. The initial stages of such a study are undertaken: a general definition of a spin group is proposed and the structures of various classes of spin groups are obtained.

Spin Groups and Galilei Invariance: The non-relativistic world has the Galilei group as a symmetry group in the sense that physical laws of such a world must be invariant under that group. Quantum mechanics is distinguished from non-quantum mechanics by its use of concepts which lack clear classical analogy; intrinsic spin is such a concept, and an electron is a particle with spin. Since non-relativistic quantum mechanics is as well understood as it is, it is peculiar that only recently has the non-relativistic analogue of the Dirac equation for the relativistic electron been considered from first principles.

This equation is re-examined in the light of the present work on general spin groups.

<u>Galilei Invariance and Clifford Algebras:</u> The real world is believed to be relativistic, however much of the phenomena we see appears to be non-relativistic. The Galilei group is, in a sense, the limit of the group describing the relativistic world. If one considers the spin groups of these groups as basic, the Galilei spin group is a limit of the non-relativistic one and this limit can be regarded as the result of a limit of the relativistic Clifford algebra to the non-relativistic one; this notion is formulated and the most obvious results proved.

ACKNOWLEDGEMENT

It is indeed a great pleasure to express my sincere gratitude to Hans Künzle who has been a help to me in so many ways. On the material side, he has provided financial support at various times, most particularly during the final stages of this work. But more importantly, his breadth of knowledge and interest, his uncompromisingly high personal standards, and intellectual honesty (and industry) are attributes in such measure as I can only hope to possess.

To Jean-François Dumais goes my thanks for his active interest in several aspects of this research.

And to all my friends of Mathematical Physics, with whom the discussions have ranged from lively to heated but never boring, thank you.

Early financial assistance was provided in part by a National Sciences and Engineering Research Council Postgraduate Scholarship and later by the University of Alberta in the form of a Dissertation Fellowship. To both of these institutions, I extend my grateful thanks.

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CHAPTER I

CLIFFORD ALGEBRAS AND SPIN GROUPS IN ABSTRACTO

I.0. INTRODUCTION

We begin section 1 of this chapter with the notions of real orthogonal spaces and abstract Clifford algebras after which several results, useful later, are proved or outlined. One of these concerns the nature of a special class of "degenerate" Clifford algebras. Section 2 deals with the identification of spin groups within a Clifford algebra context. Next, the spin analogues of the Lorentz, homogeneous Galilei, and de Sitter groups are introduced and the structure of the spin group corresponding to the degenerate space in section 1 is computed. A brief section 4 deals with Lie algebras and their appearance in the Clifford algebra scheme; those corresponding to the groups of primary importance in section 3 are presented. Some possibilities and difficulties, when the underlying orthogonal space has arbitrary degeneracy, are explored in section 5; a particular class of "more-degenerate" spin groups (some of whose members being possible space-time groups) is investigated in detail. The extended Galilei group is considered in section 6 within a Clifford algebra - spin group formulation. Finally, the chapter ends with several remarks both elucidatory and speculative in a section: Notes and References.

Of the formal propositions stated or proved in this chapter, the following are, to the best knowledge of the author, new: Thm. I.1.3, Prop. I.1.4, Prop. I.1.5, Prop. I.1.6, Prop. I.1.7; Th. I.3.1, Cor. I.3.2, Cor. I.3.3, Thm. I.3.4, Cor. I.3.5, Cor. I.3.6; Thm. I.5.1, Cor. I.5.2, Prop. I.6.1; in section 7, Thm. and Prop. (notes on §1).

I.1. REAL ORTHOGONAL SPACES AND CLIFFORD ALGEBRAS

A *real orthogonal space* is a vector space over $\mathbb{R}$ with a distinguished real quadratic form Q (or equivalently, with the symmetric real bilinear form B that corresponds to Q). Our real orthogonal spaces will be finite dimensional as real vector spaces. There is no loss in generality in assuming the bilinear form to have been diagonalized and as a result, every real orthogonal space is of the form $\mathbb{R}^{r,p,q}$, where by $\mathbb{R}^{r,p,q}$ we mean the space $\mathbb{R}^n$ $(n = p+q+r)$, with the bilinear form B, the orthogonal space being denoted $(\mathbb{R}^n, B)$:

$$B = \mathrm{diag}\,(0,\cdots,0,-1,\cdots,-1,1,\cdots,1) \qquad (1.1)$$

with r zero, p negative, and q positive entries. The form is non-degenerate when $r = 0$ and we write $\mathbb{R}^{p,q}$ for $\mathbb{R}^{0,p,q}$ (Porteous (1969), p. 149).

While most people studying these spaces make the assumption of non-degeneracy, it is exactly this hypothesis which will be relaxed in the present study. If there is a single key observation upon which much of this work is based, then it is the following very simple result.

Proposition I.1.1. The non-degenerate real orthogonal space of least dimension which contains $\mathbb{R}^{r,p,q}$ as an orthogonal subspace (orthogonal inclusions are defined below) is $\mathbb{R}^{r+p,r+q}$. Thus any such containing space is of the form $\mathbb{R}^{p',q'}$ with $p' \geq p+r$, $q' \geq q+r$.

Before getting to the proof, we recall some useful definitions (see for example Chevalley (1954)). An *orthogonal inclusion* (or *embedding*) of the orthogonal space $(\mathbb{R}^n, B)$ into the orthogonal space $(\mathbb{R}^{n'}, B')$, written, $(\mathbb{R}^n, B) \subset (\mathbb{R}^{n'}, B')$ is a mapping $i : \mathbb{R}^n \to \mathbb{R}^{n'}$, both linear and one-to-one, such that $i^*B' = B$ (as is customary, $(i^*B')(x,y) = B'(i(x),i(y))$, $\forall x,y \in \mathbb{R}^n$). The *orthogonal complement* $W^\perp$ of a subset W of $(\mathbb{R}^n, B)$ is defined by $W^\perp = \{x \in \mathbb{R}^n : B(x,w) = 0,\ \forall w \in W\}$; a subspace W of $(\mathbb{R}^n, B)$ is *totally isotropic* if $W \subset W^\perp$. The *index* ν of a real orthogonal space is the maximal dimension of a totally isotropic subspace. It is well known (Porteous (1969), p. 161) that $\nu(\mathbb{R}^{p,q}) = \min\{p,q\}$, for if $W \subset \mathbb{R}^{p,q}$ is isotropic and $p_1 : \mathbb{R}^{p,q} \to \mathbb{R}^{p,0}$, $p_2 : \mathbb{R}^{p,q} \to \mathbb{R}^{0,q}$ are the projections of $\mathbb{R}^{p,q} \cong \mathbb{R}^{p,0} \oplus \mathbb{R}^{0,q}$, then $p_1|_W$ and $p_2|_W$ are one-to-one ($p_1(w) = 0$ for some $w \in W$ implies $w \in \mathbb{R}^{0,q}$ and $B(w,w) = 0$ implies $w = 0$; likewise for $p_2|_W$). Hence $\dim W = \operatorname{rank} p_1|_W \leq p$ and also $\dim W = \operatorname{rank} p_2|_W \leq q$, implying $\nu(\mathbb{R}^{p,q}) \leq \min\{p,q\}$. Also, $\nu(\mathbb{R}^{p,q}) \geq \min\{p,q\}$ since if $e_1, \cdots, e_p, e_{p+1}, \cdots, e_{p+q}$ form an orthonormal basis ($B(e_i,e_j) = -\delta_{ij}$, $B(e_{p+m}, e_{p+n}) = \delta_{mn}$, $B(e_i,e_{p+n}) = 0$ for $1 \leq i$, $j \leq p$, $1 \leq m$, $n \leq q$) of $\mathbb{R}^{p,q}$, then the subspace W spanned by those $e_i + e_{i+\min\{p,q\}}$, $1 \leq i \leq \min\{p,q\}$, is totally isotropic of dimension $\min\{p,q\}$.

<u>Proof of Prop. I.1.1.</u>: If $\mathbb{R}^{r,p,q} \subset \mathbb{R}^{p',q'}$, then obviously $p' \geq p$ and $q' \geq q$, and from the decompositions $\mathbb{R}^{r,p,q} \cong \mathbb{R}^{r,0,0} \oplus \mathbb{R}^{p,q}$, $\mathbb{R}^{p',q'} \cong \mathbb{R}^{p'-p,q'-q} \oplus \mathbb{R}^{p,q}$ it follows that $\mathbb{R}^{r,0,0} \subset \mathbb{R}^{p'-p,q'-q}$. Thus $r \leq \nu(\mathbb{R}^{p'-p,q'-q}) = \min\{p'-p,q'-q\} \Rightarrow p' \geq r+p,\ q' \geq r+q$ so

$\mathbb{R}^{r,p,q} \subset \mathbb{R}^{p',q'} \Rightarrow p' \geq r+p$, $q' \geq r+q$ and conversely as is now obvious. □

<u>Corollary I.1.2</u>: $\nu(\mathbb{R}^{r,p,q}) = r + \min\{p,q\}$.

<u>Proof</u>: Mimic the argument for $\nu(\mathbb{R}^{p',q'})$, using

$$\mathbb{R}^{r,p,q} \cong \mathbb{R}^{r,0,0} \oplus \mathbb{R}^{p,0} \oplus \mathbb{R}^{0,q} \quad .$$ □

Probably the simplest physically relevant example of Prop. I.1.1 is the orthogonal inclusion $\mathbb{R}^{1,0,3} \overset{i}{\subset} \mathbb{R}^{1,4}$; if $\{e_\alpha : 0 \leq \alpha \leq 3\}$, $\{f_i : 0 \leq i \leq 4\}$ are orthonormal bases of $\mathbb{R}^{1,0,3}$, $\mathbb{R}^{1,4}$ respectively (i.e. $B(e_0,e_0) = 0$, $B(e_0,e_A) = 0$, $B(e_A,e_B) = \delta_{AB}$ and $B'(f_0,f_0) = -1$, $B'(f_A,f_B) = \delta_{AB}$, $B'(f_4,f_4) = 1$, $B'(f_0,f_A) = B'(f_0,f_4) = B'(f_A,f_4) = 0$ for $1 \leq A,B \leq 3$) then $i(e_0) = f_0+f_4$, $i(e_A) = f_A$, $1 \leq A \leq 3$ up to a basis change.

A *real Clifford algebra* for the real orthogonal space (X,B) is a real associative algebra, denoted $C(X,B)$, $C(X)$ (and C) if B (and X) is (are) understood, generated as a ring by $\mathbb{R}$ and X and subject to the condition that for $x \in X$:

$$x^2 + B(x,x)\cdot 1 = 0 \tag{1.2}$$

where 1 = unity in $C(X)$ is identified with $1 \in \mathbb{R}$, and may therefore be omitted. Additionally, $C(X)$ is to contain isomorphic copies of $\mathbb{R}$, X as linear subspaces (see Notes for references on Clifford algebras).

The question of existence is settled by the following construction. Let $\mathcal{T}$ be the covariant (or contravariant) tensor algebra of X and let $\mathcal{J}$ be the (two-sided) ideal generated by the elements $x \otimes x + B(x,x)$, $\forall\, x \in X$. The quotient $\mathcal{A} = \mathcal{T}/\mathcal{J}$ is a Clifford algebra for (X,B) and $\dim_{\mathbb{R}} \mathcal{A} = 2^n$, where $n = \dim_{\mathbb{R}} X$.

An *orthonormal subset* (Porteous (1969), p. 242) of C (or X) is a linearly independent set $\{\gamma^i\}$, $\gamma^i \in C$ (or X) satisfying (with $B = (B^{ij}) = \mathrm{diag}(0,\cdots,0,-1,\cdots,-1,1,\cdots,1)$):

$$\gamma^i\gamma^j + \gamma^j\gamma^i = -2\cdot B(\gamma^i,\gamma^j) = -2\cdot B^{ij} \tag{1.3}$$

For reasons involving the physical interpretation of $B = (B^{ij})$ when $r = 1$, $p = 0$, $q = 3$, we label the γ^i 's , and B^{ij} in the upper position (leading to the identification of $\{\gamma^i\}$ with a set of 1-forms on space-time). The orthonormal subset is of *type* (r,p,q) if in $\{(\gamma^i)^2\}$ there are r zeros, p positive ones and q negative ones; when $r = 0$ the type is (p,q) . When $X = \mathbb{R}^{r,p,q}$ with $n = r+p+q$ we identify X with $\mathrm{span}_{\mathbb{R}}\, \{\gamma^i : 1 \leq i \leq n\}$.

The following facts are standard (see: Notes and References at the chapter's end):

(i) If C is a Clifford algebra for an n - dimensional orthogonal space, then $\dim C \leq 2^n$.

(ii) All 2^n - dimensional Clifford algebras, for an n - dimensional orthogonal space, are isomorphic.

(iii) If C is a Clifford algebra for the n - dimensional non-degenerate orthogonal space $R^{p,q}$, $n = p+q$, then $\dim C = 2^n$ or 2^{n-1} , the second case being possible only if $p-q-1 \equiv 0 \pmod 4$ (in fact when n is odd and $\gamma^1\gamma^2\cdots\gamma^n = \pm 1$ for any orthonormal basis $\{\gamma^i\}$ of $R^{p,q}$).

On account of (ii), the unique-up-to-isomorphism Clifford algebra of dimension 2^n for X of dimension n , is referred to, by Porteous (1969, p. 245) as the *universal* Clifford algebra for X . When $(X,B) = R^{r,p,q}$, this universal Clifford algebra is denoted $R_{r,p,q}$, and when $(X,B) = R^{p,q}$ we abbreviate $R_{0,p,q}$ to $R_{p,q}$.

We generalize (iii) to a class of degenerate orthogonal spaces in the following (see: Notes and References).

Theorem I.1.3.: Let $\gamma^0,\gamma^1,\cdots,\gamma^n$ be an orthonormal basis of $R^{1,p,q}$ with $(\gamma^0)^2 = 0$ and $n = p+q$. If C is the Clifford algebra generated by the $\{\gamma^i\}$, written $C = \langle 1,\gamma^i : 0\leq i\leq n\rangle$, then:

(i) $\dim C = 2^{n+1}$ if n is even or if n is odd and γ^0 , $\gamma^0\gamma^1\cdots\gamma^n$ are linearly independent.

(ii) $\dim C = 3\times 2^{n-1}$ if n is odd and γ^0 , $\gamma^0\gamma^1\cdots\gamma^n$ are linearly dependent; a necessary condition for this is that $p-q-1 \equiv 0 \pmod 4$.

Before proceeding to the proof, which is both tricky and tedious, we recall some concepts to be used both in the proof and later on.

For $X = \mathbb{R}^{r,p,q}$, the orthogonal involution $-1_X : X \to X$ defined by, $x \to -x$, induces on $\mathbb{R}_{r,p,q}$ the *main involution* $^\wedge$, defined as follows (Porteous (1969), p. 252):

$$1^\wedge = 1 \ , \ (\gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_m})^\wedge = (-1)^m \gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_m} \tag{1.4}$$

with $\{\gamma^i\}$ an orthonormal basis of X generating $C(X) = \mathbb{R}_{r,p,q}$, and $i_1<i_2<\cdots<i_m$. The main involution provides a direct sum decomposition $C(X) = \overset{+}{C}(X) \oplus \overset{-}{C}(X)$, where $\overset{\pm}{C}(X) = \{a \in C(X) : a^\wedge = \pm a\}$ respectively are the sets of *even* (+) and *odd* (-) elements. $\overset{+}{C}(X)$ is, itself, often a Clifford algebra as we now indicate.

Proposition I.1.4.: Assuming $p+q \geq 1$,

$$\overset{+}{\mathbb{R}}_{r,p,q} \cong \begin{cases} \mathbb{R}_{r,p,q-1} & \text{when} \quad q \geq 1 \\ \\ \mathbb{R}_{r,q,p-1} & \text{when} \quad p \geq 1 \end{cases}$$

See Notes for the cases $p=q=0$, that is, when $X = \mathbb{R}^r$ with the zero bilinear form: $\mathbb{R}_{r,0,0} \cong$ exterior algebra on $\mathbb{R}^r$.

Proof of Prop. I.1.4.: Let $\{\gamma^i\}$ be an orthonormal basis of $\mathbb{R}^{r,p,q}$ which generates $\mathbb{R}_{r,p,q}$. The real algebra $\overset{+}{\mathbb{R}}_{r,p,q}$ is spanned by products of an even number of the γ^i's. When $q \geq 1$, choose a fixed γ^k such that $(\gamma^k)^2 = -1$ and notice that $\overset{+}{\mathbb{R}}_{r,p,q} = \langle 1 , \gamma^i\gamma^k : i \neq k\rangle$; moreover $\{\gamma^i\gamma^k : i \neq k\}$ is an orthonormal subset of type $(r,p,q-1)$ and $\mathbb{R}^{r,p,q-1} \cong \text{span}_{\mathbb{R}} \{\gamma^i\gamma^k : i \neq k\}$, hence $\overset{+}{\mathbb{R}}_{r,p,q} \cong \mathbb{R}_{r,p,q-1}$. The argument is similar when $p \geq 1$, except that for fixed γ^k such that $(\gamma^k)^2 = 1$, $\{\gamma^i\gamma^k : i \neq k\}$ is an orthonormal subset of type $(r,q,p-1)$.

Notice that when both $p,q \geq 1$, $\mathbb{R}_{r,p,q-1} \cong \mathbb{R}_{r,q,p-1}$. □

In the next two paragraphs: $X = \mathbb{R}^{r,p,q}$ and $C(X) = \mathbb{R}_{r,p,q}$.

The orthogonal involution -1_X of X also induces on $C(X)$ the *conjugation anti-involution* $^-$ defined (Porteous (1969), p. 252) by:

$$1^- = 1 \ , \ (\gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_m})^- = (-1)^m \gamma^{i_m}\cdots\gamma^{i_2}\gamma^{i_1} \tag{1.5}$$

with the i_k 's and γ^{i_k} 's as in (1.4). Note also that

$$(\gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_m})^- = (-1)^{\frac{m^2+m}{2}} \gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_m} \ .$$

Finally the orthogonal involution 1_X of X induces on $C(X)$ the *reversion anti-involution* (Porteous (1969), p. 252):

$$1 \to 1 \ , \ \gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_m} \to \gamma^{i_m}\cdots\gamma^{i_2}\gamma^{i_1} = (-1)^{\frac{m^2-m}{2}} \gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_m} \tag{1.6}$$

again with the γ^{i_k} 's and indices as in (1.4).

We turn now to the proof of Theorem I.1.3.

<u>Proof of Theorem I.1.3.</u>: The following notation will be used here and later:

(i) I,J denote subsets of $\{0,1,\cdots,n\}$ and I',J' their complements.

(ii) γ^I will denote $\gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_m}$ or 1 depending whether $I = \{i_1,i_2,\cdots,i_m\}$ with $i_1 < i_2 < \cdots < i_m$ or $I = \phi$; $|I|$ = card I .

(iii) $\sum_I a_I \cdot \gamma^I$ denotes a sum over all subsets of $\{0,1,\cdots,n\}$.

(iv) for all $a,b \in C$: $[a,b] = ab-ba$, $\{a,b\} = ab+ba$

(v) A,B etc. will be indices $1 \leq A$, $B \leq n$; A' the complement of $\{A\}$ in $\{0,1,\cdots,n\}$

(vi) $\overset{\pm}{P}(S)$ denotes the family of subsets of $S \subset \{0,1,\ldots,n\}$ of even/odd cardinality respectively.

Supposing $\sum_I a_I \cdot \gamma^I = 0$, we show that all $a_I = 0$ when n is even or when n is odd and $\gamma^0, \gamma^0\gamma^1 \cdots \gamma^n$ are linearly independent, and that otherwise there are 2^{n-1} freely assignable a_I 's.

If for fixed A we conjugate $\sum_I a_I \cdot \gamma^I = 0$ by γ^A (i.e. simplify $\sum_I a_I \cdot \gamma^A \gamma^I (\gamma^A)^{-1} = 0$) and use $\sum_I a_I \cdot \gamma^I = 0$, then:

$$\sum_{[A,I]=0} a_I \cdot \gamma^I = 0 \quad , \quad \sum_{\{A,I\}=0} a_I \cdot \gamma^I = 0 \tag{1.7}$$

where $\sum\limits_{[A,I]=0}$ is the sum over all I for which γ^A and γ^I commute ($[\gamma^A,\gamma^I]=0$) , and analogously for $\sum\limits_{\{A,I\}=0}$. Picking another fixed $B \neq A$ and conjugating (1.7) by γ^B we have:

$$\sum_{[AB,I]=0} a_I \cdot \gamma^I = 0 \quad , \quad \sum_{\{AB,I\}=0} a_I \cdot \gamma^I = 0 \tag{1.8}$$

where now $\sum\limits_{\{AB,I\}=0}$ is a sum over all I such that $\{\gamma^A,\gamma^I\} = 0$ and $\{\gamma^B,\gamma^I\} = 0$. Repeat this process for all indices $\{1,2,\cdots,n\}$ until one obtains:

$$\sum_{[1\,2\cdots n,I]=0} a_I\cdot\gamma^I = 0 \quad , \quad \sum_{\{1\,2\cdots n,I\}=0} a_I\cdot\gamma^I = 0 \tag{1.9}$$

with the obvious meaning for $\sum_{[1\,2\cdots n,I]=0}$ etc.

Now $[\gamma^A,\gamma^I] = 0$ for all $A \varepsilon \{1,2,\ldots,n\} \Rightarrow I \varepsilon \overset{+}{P}(A')$ or $\{A\} \subset I \varepsilon \bar{P}(\{0,1,\ldots,n\})$ for all $A \varepsilon \{1,2,\ldots,n\}$ and two possibilities arise depending on the parity of $|I|$, namely:

$$I \,\varepsilon \bigcap_{1\le A\le n} \overset{+}{P}(A') = \{\phi\} \qquad \text{when} \quad |I| \text{ even}$$

$$I \supset \bigcup_{1\le A\le n} \{A\} = \{1,2,\ldots,n\} \qquad \text{when} \quad |I| \text{ odd.}$$

A moment's thought then shows:

$$\gamma^I = 1 \qquad |I| \text{ even} \tag{1.10a}$$

$$\gamma^I = \begin{cases} \gamma^0\gamma^1\cdots\gamma^n & |I| \text{ odd, } n \text{ even} \\ \gamma^1\gamma^2\cdots\gamma^n & |I| \text{ odd, } n \text{ odd} \end{cases} \tag{1.10b}$$

Further, $\{\gamma^A,\gamma^I\} = 0$, $1\le A\le n \Rightarrow \{A\} \subset I \varepsilon \overset{+}{P}(\{0,1,\ldots,n\})$ or $I \varepsilon \bar{P}(A')$ for all $A \varepsilon \{1,2,\ldots,n\}$ and therefore much as before:

$$\gamma^I = \begin{cases} \gamma^1\gamma^2\cdots\gamma^n & |I| \text{ even, } n \text{ even} \\ \gamma^0\gamma^1\cdots\gamma^n & |I| \text{ even, } n \text{ odd} \end{cases} \tag{1.11a}$$

$$\gamma^I = \gamma^0 \qquad |I| \text{ odd} \tag{1.11b}$$

Now using (1.10), (1.11) to rewrite (1.9):

$$a_\phi\cdot 1 + a_{0\,1\cdots n}\cdot \gamma^0\gamma^1\cdots\gamma^n = 0 \; , \; a_0\cdot\gamma^0 + a_{1\,2\cdots n}\cdot\gamma^1\gamma^2\cdots\gamma^n = 0 \tag{1.12a}$$

when n is even; and for n odd:

$$a_\phi\cdot 1 + a_{1\,2\cdots n}\cdot\gamma^1\gamma^2\cdots\gamma^n = 0\ ,\ a_0\cdot\gamma^0 + a_{0\,1\cdots n}\cdot\gamma^0\gamma^1\cdots\gamma^n = 0 \quad (1.12b)$$

Thus $\sum_I a_I\cdot\gamma^I = 0 \Rightarrow (1.12)$. However for $0 \notin J$ (J fixed), we also have $0 = \sum_I a_I\cdot\gamma^I\gamma^J = \sum_I b_I\cdot\gamma^I$ where the b_I 's are a rearrangement up-to-a-sign of the a_I 's. Consequently we have relations on the b_I's analogous to (1.12): merely replace the a_I's by identically indexed b_I 's. Precisely, $\gamma^I\gamma^J = \mu(I,J)\cdot\gamma^{I\Delta J}$ with $\mu(I,J) = \pm 1$ (see Notes), and $I\Delta J = (I\cap J')\cup(J\cap I')$ = symmetric difference of I,J . Now it's easily shown that given I , there is a unique $\tilde{I} \subset \{0,1,\cdots,n\}$ such that $I = \tilde{I}\Delta J$, hence $b_I = a_{\tilde{I}}\,\mu(\tilde{I},J)$. Of course, $\tilde{I}$ implicitly depends also on J , so different J 's give different analogues of (1.12). For fixed J , the expressions analogous to (1.12) are:

$$\begin{aligned} &a_J\,\mu(J,J)\cdot 1 + a_{J'}\,\mu(J',J)\cdot\gamma^0\gamma^1\cdots\gamma^n = 0 \\ &a_{\{0\}\cup J}\,\mu(\{0\}\cup J,J)\cdot\gamma^0 + a_{J'\setminus\{0\}}\,\mu(J'\setminus\{0\},J)\cdot\gamma^1\gamma^2\cdots\gamma^n = 0 \end{aligned} \quad (1.13a)$$

where n is even, and

$$\begin{aligned} &a_J\,\mu(J,J)\cdot 1 + a_{J'\setminus\{0\}}\,\mu(J'\setminus\{0\},J)\cdot\gamma^1\gamma^2\cdots\gamma^n = 0 \\ &a_{\{0\}\cup J}\,\mu(\{0\}\cup J,J)\cdot\gamma^0 + a_{J'}\,\mu(J',J)\cdot\gamma^0\gamma^1\cdots\gamma^n = 0 \end{aligned} \quad (1.13b)$$

when n is odd.

For n even, from (1.13a) multiply by γ^0 to show $a_J = 0$, hence $a_{J'} = 0$ as well; thus n even $\Rightarrow$ all $a_I = 0$, $\forall I$.

For n odd, from (1.13b), assuming first that γ^0, $\gamma^0\gamma^1\cdots\gamma^n$ are linearly independent $(\Rightarrow \gamma^1\gamma^2\cdots\gamma^n \notin R)$, we see that $a_J = 0$, $a_{\{0\}\cup J} = 0$, $\forall J \subset \{1,2,\cdots,n\}$; so in this case, $a_I = 0$, $\forall I$. Finally for γ^0, $\gamma^0\gamma^1\cdots\gamma^n$ linearly dependent and n odd, $\gamma^1\gamma^2\cdots\gamma^n \notin R$ (for if $\gamma^1\gamma^2\cdots\gamma^n \in R$, then $0 = [\gamma^0,\gamma^1\gamma^2\cdots\gamma^n] = \gamma^0\gamma^1\cdots\gamma^n - \gamma^1\cdots\gamma^n\gamma^0 = 2\gamma^0\gamma^1\cdots\gamma^n$ so $\gamma^0 = 0$ contrary to the hypotheses), hence $a_J = 0$, $a_{J'\setminus\{0\}} = 0$ for all $J \subset \{1,2,\cdots,n\}$. Consequently, the only possible non-trivial relations are 2^{n-1} in number, namely:

$$a_{\{0\}\cup J}\,\mu(\{0\}\cup J,J)\cdot\gamma^0 + a_{J'}\,\mu(J',J)\cdot\gamma^0\gamma^1\cdots\gamma^n = 0\ ,\ \forall\ J \subset \{1,2,\cdots,n\}\ .$$

Furthermore, if $\gamma^0\gamma^1\cdots\gamma^n = \lambda\cdot\gamma^0$ for some $\lambda \in R$, $\lambda \neq 0$, then using the fact that $(\gamma^1\gamma^2\cdots\gamma^n)^2 = (-1)^{\frac{n(n-1)}{2}+q}$, we find $\lambda^2 = (-1)^{\frac{n(n-1)}{2}+q}$ and $\frac{n(n-1)}{2} + q \equiv 0 \pmod 2 \Rightarrow p-q-1 \equiv 0 \pmod 4$. This completes the proof. $\square$

Before closing this section we state a number of results on $R_{r,p,q}$, for arbitrary r; certain of these will be referred to later.

<u>Proposition I.1.5.</u>: If $a \in R_{r,p,q}$, and if for all $x \in R^{r,p,q}$, $ax = x\hat{a}$, then $a \in R_{r,0,0}$ and conversely $(R_{r,0,0} = \langle 1,\gamma^1,\cdots,\gamma^r\rangle$, where $\gamma^1,\cdots,\gamma^r$ are those members of the orthonormal basis $\{\gamma^1,\gamma^2,\cdots,\gamma^n\}$, $n = r+p+q$, whose squares vanish; see Notes).

Proof: The converse is almost trivial: for $a \in \langle 1,\gamma^1,\cdots,\gamma^r\rangle$, show that $a\gamma^i = \gamma^i a^\wedge$, $1 \leq i \leq n$ (which follows easily using the definition of $^\wedge$). In the forward direction, suppose $a = a_+ + a_-$, where $a_\pm \in \overset{\pm}{R}_{r,p,q}$. Then $ax = xa^\wedge$, $\forall\, x \in R^{r,p,q} \iff a_\pm x = \pm x a_\pm$, $\forall\, x \in R^{r,p,q}$. If $a_+ = \sum_{|I|\text{even}} a_{+I}\cdot\gamma^I$, then $a_+\gamma^i = \gamma^i a_+$ when $(\gamma^i)^2 = \pm 1$, implies that $a_{+I} = 0$ whenever $i \in I$. Repeating this for all such γ^i, one finds $a_+ \in \langle 1,\gamma^1,\cdots,\gamma^r\rangle$. Next, if $a_- = \sum_{|I|\text{odd}} a_{-I}\cdot\gamma^I$, then $a_-\gamma^i = -\gamma^i a_-$ for $r+1 \leq i \leq n$, implies $a_{-I} = 0$ whenever $i \in I$; and repeating for all such i, $a_- \in \langle 1,\gamma^1,\cdots,\gamma^r\rangle$. Thus $a = a_+ + a_- \in \langle 1,\gamma^1,\cdots,\gamma^r\rangle$. □

Proposition I.1.6.: An element $a \in R_{r,0,0}$ is invertible if and only if $\mathrm{Re}(a) \neq 0$ (where $\mathrm{Re}(a) = a_\phi$ in the expansion $a = \sum_{I\subset\{1,\cdots,r\}} a_I\cdot\gamma^I$).

Proof: Simply notice that $(a-a_\phi)^{r+1} = 0$, that is, $a-a_\phi$ is nilpotent, from which $a_\phi \neq 0 \Rightarrow a^{-1}$ exists. □

See the Notes and Comments for further remarks related to Proposition I.1.6.

Proposition I.1.7.: If $Z(R_{r,p,q})$ denotes the centre of $R_{r,p,q}$ then, with the notation of Proposition I.1.5, $n = r+p+q$:

$$Z(R_{r,p,q}) = \begin{cases} \overset{+}{R}_{r,0,0} & n \text{ even} \\ \overset{+}{R}_{r,0,0} + R\cdot\gamma^1\gamma^2\cdots\gamma^n & n \text{ odd} \end{cases}$$

and when $p+q \geq 1$:

$$Z(\overset{+}{R}_{r,p,q}) = \begin{cases} \overset{+}{R}_{r,0,0} + R\cdot\gamma^1\gamma^2\cdots\gamma^n & n \text{ even} \\ \overset{+}{R}_{r,0,0} & n \text{ odd} \end{cases} .$$

<u>Proof</u>: Let $\alpha = \sum_I \alpha_I\cdot\gamma^I \in Z(\overset{+}{R}_{r,p,q})$. Requiring that $\alpha\gamma^i = \gamma^i\alpha$, $1 \leq i \leq r$ implies nothing when $|I|$ even, and $\alpha_I \neq 0$ possible only if $\{1,2,\cdots,r\} \subset I$, for $|I|$ odd. Now $\alpha\gamma^i = \gamma^i\alpha$, $r+1 \leq i \leq n$ implies $\alpha_I = 0$ if $I \not\subset \{1,\cdots,r\}$ when $|I|$ is even, and $\alpha_I \neq 0$ only if $\{r+1,\cdots,n\} \subset I$ when $|I|$ odd. Thus, $|I|$ even => $\alpha_I = 0$ unless $I \subset \{1,\cdots,r\}$, and $|I|$ odd => $\alpha_I = 0$ unless $\{1,2,\cdots,n\} \subset I$, which is only possible for n odd. This proves the first part. The second part follows by an argument employing Prop. I.1.4. □

I.2. SPIN GROUPS

Before recalling the definition of a spin group for a non-degenerate orthogonal space, we introduce a couple of definitions.

Suppose $X = \mathbb{R}^{r,p,q}$, with $C(X)$ denoting the universal Clifford algebra $\mathbb{R}_{r,p,q}$.

The *norm* on $C(X)$ is the mapping: $N : C(X) \to C(X)$ by $a \to a^{-}a$, $^{-}$ denoting conjugation on $C(X)$. (See Porteous (1969), p. 260.)

The *Clifford group* (Porteous (1969), p. 254) for $X = \mathbb{R}^{r,p,q}$ is: $\Gamma(X) = \{g \in C(X) : g \text{ is invertible; } g^{-1}x\,g^{\wedge} \in X, \forall\, x \in X\}$. When $p+q \geq 1$, the *even (special) Clifford group* $\Gamma^{+}(X)$ is defined by $\Gamma^{+}(X) = \Gamma(X) \cap C^{+}(X)$. That $\Gamma(X)$ is a group follows from the next well known result.

Letting ρ_X (or just ρ, X being understood), denote the mapping (see Notes) $\rho_X : \Gamma(X) \to \mathrm{End}_{\mathbb{R}}(X)$, where $\rho_X(g)$ acts to the right by $x \to x\cdot\rho_X(g) = g^{-1}x\,g^{\wedge}$, we have for $g,g' \in \Gamma(X)$, $\rho_X(gg') = \rho_X(g)\rho_X(g')$. Note also that $\rho_X(g)$ is an isomorphism, in fact an orthogonal automorphism of X. ($\rho_X(g)$ is one-to-one, hence onto, because $x\cdot\rho_X(g) = 0 \Rightarrow g^{-1}x\,g^{\wedge} = 0 \Rightarrow x = 0$; $\rho_X(g)$ is orthogonal if $B(x\cdot\rho_X(g), y\cdot\rho_X(g)) = B(x,y)$ for all $g \in \Gamma(X)$, $x,y \in X$. But it suffices by the polarization identity, $B(x,y) = \frac{1}{4}\{B(x+y,x+y)-B(x-y,x-y)\}$, to show only that $B(x\cdot\rho_X(g), x\cdot\rho_X(g)) = B(x,x)$ for $g \in \Gamma(X)$, $x \in X$. This follows because $B(x,x) = -x^2 = x^{\wedge}x$, and $(x\cdot\rho_X(g))^{\wedge}(x\cdot\rho_X(g)) = x^{\wedge}x$ for all $x \in X$.).

Consequently, ρ_X is a group homomorphism: $\rho_X \in \mathrm{Hom}(\Gamma(X), O(X))$, where $O(X)$ is the orthogonal group of the orthogonal space

$(X,B) = \mathbb{R}^{r,p,q}$; that is $O(X) = \{$linear isomorphisms $x \to x'$ of X, such that $B(x',y') = B(x,y)\}$. Using Prop. I.1.5, I.1.6 we easily see that $\ker \rho_X = \{$invertibles in $\mathbb{R}_{r,0,0}\} = \{a \in \mathbb{R}_{r,0,0} : \mathrm{Re}(a) \neq 0\}$.

When (X,B) is non-degenerate $(X = \mathbb{R}^{p,q})$, one has the standard definitions of *pin group* and *spin group* (Chevalley (1954), p. 52; Dieudonné (1963), p. 55):

$$\mathrm{Pin}(X) = \{g \in \Gamma(X) : N(g) \in \{-1,1\}\}$$

$$\mathrm{Spin}(X) = \{g \in \overset{+}{\Gamma}(X) : N(g) \in \{-1,1\}\} = \mathrm{Pin}(X) \cap \overset{+}{C}(X) \quad .$$

In this non-degenerate situation $\mathrm{Pin}(X)$, $\mathrm{Spin}(X)$ are subgroups of $\Gamma(X)$, $\overset{+}{\Gamma}(X)$ respectively, and if ρ_X also denotes the restriction of ρ_X to $\mathrm{Pin}(X)$, $\mathrm{Spin}(X)$ respectively, we have (see remarks in the previous paragraph): $\ker \rho_X = \{-1,1\}$ and ρ_X surjective (that is $\rho_X(\mathrm{Pin}(X)) = O(X)$ and $\rho_X(\mathrm{Spin}(X)) = SO(X)$ = subgroup of $O(X)$ which preserves orientations $= O(X) \cap \det^{-1}\{1\}$, $\det$ denoting determinant). Thus, one has the group isomorphisms (for non-degenerate X, see Porteous (1969), Prop. 13.48):

$$\mathrm{Pin}(X) / \{-1,1\} \cong O(X)$$

$$\mathrm{Spin}(X) / \{-1,1\} \cong SO(X) \quad .$$

The following notational abuse will, at times, be convenient: $\mathrm{Pin}(O(X))$ for $\mathrm{Pin}(X)$; $\mathrm{Spin}(SO(X))$ for $\mathrm{Spin}(X)$ and possible variants thereof, whose meanings will be clear from the context (e.g. $SO(r,p,q)$ means $SO(\mathbb{R}^{r,p,q})$ etc.).

In many instances, it's only the connected identity component of $SO(X)$ that is of interest. For non-degenerate X, ρ_X maps $Spin(X)$ onto $SO(X)$ hence the connected component of the identity of $Spin(X)$ onto that of $SO(X)$. As defined, $Spin(X)$ has a subgroup $Spin^+(X)$, $Spin^+(X) = \{ g \in Spin(X) : N(g) = 1\}$, whose image under ρ_X is a subgroup $SO^+(X)$ of $SO(X)$ which preserves the semi-orientations of X (Porteous (1969), p. 161, p. 268); that is, identifying $X = R^{p,q}$ with $R^p \times R^q$ and $\Lambda \in SO(X)$ with the $(p+q)\times(p+q)$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a : R^p \to R^p$, $b : R^p \to R^q$, $c : R^q \to R^p$, $d : R^q \to R^q$, we say Λ *preserves the semi-orientations of* $R^{p,q}$ if a,d preserve the orientations of R^p, R^q respectively). In fact, except for $Spin^+(X)$ when $X = R^{0,0}$, $R^{1,0}$, $R^{0,1}$, $R^{1,1}$, $Spin^+(X)$ and $SO^+(X)$ are the connected identity components of $Spin(X)$ and $SO(X)$ respectively (see Notes).

Definitions of $Pin(X)$, $Spin(X)$ for degenerate spaces X do not appear to have been considered, probably because the groups $O(X)$, $SO(X)$ do not (frequently) appear either. The definitions to be used in this thesis are exactly those given previously in the non-degenerate case. Traditionally, the definitions (c.f. Porteous (1969), p. 264) given for non-degenerate X require $Pin(X)$, $Spin(X)$ to be quotient (rather than sub-) groups of $\Gamma(X)$, $\overset{+}{\Gamma}(X)$ respectively, but the two definitions coincide when X is non-degenerate. Our definition, while being a natural generalization of the former one, has as a benefit that calculations are possible. Moreover, it yields satisfactory results for a special class of degenerate spaces which includes most of those presently of interest in physics.

I.3. PARTICULAR SPIN GROUPS

The *proper Lorentz group* (or simply Lorentz group) is the group $L_6 = SO^+(1,3) = SO^+(R^{1,3})$, and the *proper de Sitter group* is $S_{10} = SO^+(1,4) = SO^+(R^{1,4})$. The *homogeneous Galilei group* G_6 is usually defined as the semi-direct product $R^3 \circledS SO(3)$, where the action of $SO(3)$ on R^3 is the linear one (the group multiplication law is as follows: $(\vec{b},R)(\vec{b}',R') = (\vec{b}+R\vec{b}',RR')$, for $\vec{b},\vec{b}$ belonging to R^3 and R,R' being rotations of R^3) , and as such is isomorphic to the group of Euclidean motions on R^3 (for general semi-direct products see Notes, section 6).

It probably is not obvious that the homogeneous Galilei group is an orthogonal group. We describe the relationship now.

Let $\gamma = (\gamma^{ij})$ denote the bilinear form on R^4 defined by: $\gamma^{0j} = 0$, $\gamma^{AB} = \delta^{AB}$, $0 \leq j \leq 3$, $1 \leq A,B \leq 3$. (Potentially there is confusion with the notation as it appears in the proof of Thm. I.1.3; that notation, however, does not appear in the statements of results, but only in proofs and care will be taken to avoid misinterpretations). Thus $(R^4,\gamma) = R^{1,0,3}$, and we now describe $O(\gamma) = O(R^{1,0,3})$ and $SO(\gamma) = SO(R^{1,0,3})$ etc. In fact it follows very easily that:

$O(\gamma) = \{\Lambda \in GL(4;R) : \Lambda\gamma\Lambda^t = \gamma\}$, ^{t} denotes transpose

and γ is regarded as the matrix $\begin{pmatrix} 0 & \\ & 1_3 \end{pmatrix}$.

Hence:

$$O(\gamma) = \left\{ \begin{pmatrix} a & 0 \\ \vec{v} & R \end{pmatrix} : 0 \neq a \in R , \vec{v} \in R^3 , R \in O(3) \right\}$$

$$SO(\gamma) = \left\{ \begin{pmatrix} a & 0 \\ \vec{v} & R \end{pmatrix} : a = \det R \ , \ \vec{v} \in R^3 \ , \ R \in O(3) \right\}$$

$$SO^+(\gamma) = \left\{ \begin{pmatrix} 1 & 0 \\ \vec{v} & R \end{pmatrix} : \vec{v} \in R^3 \ , \ R \in SO(3) \right\} .$$

The identification $R^3 \circledS SO(3) \to SO^+(R^{1,0,3})$ by $(\vec{v},R) \to \begin{pmatrix} 1 & 0 \\ \vec{v} & R \end{pmatrix}$ establishes the orthogonal nature of the homogeneous Galilei group; from now on, $SO^+(R^{1,0,3})$, $SO^+(\gamma)$, $SO^+(1,0,3)$ etc. will be used to denote the homogeneous Galilei group (for a definition of $SO^+(X)$, for arbitrary X , see section 4.)

While it is obvious that $SO^+(1,3)$ is a subgroup of $SO^+(1,4)$, it is perhaps less so that $SO^+(1,0,3)$ is also. In retrospect, one might be tempted to say this comes about because (as remarked upon in section 1) $R^{1,0,3} \subset R^{1,4}$; indeed also $R^{1,3} \subset R^{1,4}$ and in fact $R^{1,4}$ is the "smallest" non-degenerate orthogonal space containing both $R^{1,0,3}$ and $R^{1,3}$ orthogonally.

To be specific, let γ^i , $0 \leq i \leq 4$ be an orthonormal basis of $R^{1,4}$, with $g = (g^{ij}) = \mathrm{diag}(-1,1,1,1,1)$ and suppose γ^i is represented as a row vector: 0 in all columns except 1 in the i 'th, $0 \leq i \leq 4$. By explicit calculation (see the Notes):

$$\begin{aligned} SO^+(1,3) \cong SO^+(1,4)_{\gamma^4} &= \text{stabilizer of } \gamma^4 \text{ in } SO^+(1,4) \\ &= \{T \in SO^+(1,4) : \gamma^4 T = \gamma^4\} \end{aligned} \tag{3.1}$$

$$\begin{aligned} SO^+(1,0,3) \cong SO^+(1,4)_{\gamma^0+\gamma^4} &= \text{stabilizer of } \gamma^0+\gamma^4 \text{ in } SO^+(1,4) \\ &= \{T \in SO^+(1,4) : (\gamma^0+\gamma^4)T = \gamma^0+\gamma^4\} \end{aligned} \tag{3.2}$$

This is hardly surprising to one familiar with the theory of induced representations: γ^4 is a "space"-like and $\gamma^0+\gamma^4$ a null de Sitter vector.

The "most natural thing in the world" at this stage is to expect the spin group analogues to follow similarly, and they do:

$$\begin{aligned} \mathrm{Spin}^+(1,3) \cong \mathrm{Spin}^+(1,4)_{\gamma^4} &= \text{stabilizer of } \gamma^4 \text{ in } \mathrm{Spin}^+(1,4) \\ &= \{t \in \mathrm{Spin}^+(1,4) : t^{-1}\gamma^4 t = \gamma^4\} \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathrm{Spin}^+(1,0,3) \cong \mathrm{Spin}^+(1,4)_{\gamma^0+\gamma^4} &= \text{stabilizer of } \gamma^0+\gamma^4 \text{ in } \mathrm{Spin}^+(1,4) \\ &= \{t \in \mathrm{Spin}^+(1,4) : t^{-1}(\gamma^0+\gamma^4)t \\ &= \gamma^0+\gamma^4\} \end{aligned} \tag{3.4}$$

These follow most simply from the matrix representations of the spin groups given in the next chapter.

We now state and prove a structure theorem for a special class of spin groups: namely those for the degenerate orthogonal spaces $\mathbb{R}^{1,p,q}$ (c.f. Thm. I.1.3 for the Clifford algebra situation).

Theorem I.3.1.: $\mathrm{Spin}(1,p,q) = K\cdot H$ where $H \cong \mathrm{Spin}(p,q)$, $K \cong \mathbb{R}^n$, $n = p+q \geq 1$. Furthermore $H \cap K = \{1\}$, $K \lhd \mathrm{Spin}(1,p,q)$ (K is a normal subgroup), $H \cap z(K) = \{-1,1\}$ ($z(K)$ = centralizer of $K = \{g \in \mathrm{Spin}(1,p,q) : gk = kg \ , k \in K\}$) and consequently

$$\mathrm{Spin}(1,p,q)/\{-1,1\} \cong \mathbb{R}^n \circledS (\mathrm{Spin}(p,q)/\{-1,1\}) \cong \mathbb{R}^n \circledS SO(p,q)$$

$\{1\} \to \{-1,1\} \to \mathrm{Spin}(1,p,q) \xrightarrow{\rho} \mathbb{R}^n Ⓢ SO(p,q) \to \{1\}$ is an exact sequence. Before proceeding with the proof we recall the result (Jansen and Boon (1967), p. 75, ex. 31) hinted at in the theorem's statement (see Notes, section 6, for comments):

Lemma: Given a group G with subgroups H,K such that (i) $G = K\cdot H$ (ii) $H \cap K = \{1\}$ (iii) $K \lhd G$ (K normal in G), then $z(K)\cap H \lhd G$ and $G/z(K)\cap H \cong K Ⓢ (H/z(K)\cap H)$, where $z(K) = \{g \in G \; ; \; gk = kg \; , \; k \in K\}$ = centralizer of K in G.

Proof of Thm. I.3.1.: The notation will be that of Thm. I.1.3, except that $\{\gamma^i\}$, $0 \leq i \leq n$, will generate the universal Clifford algebra $\mathbb{R}_{1,p,q}$ of the orthogonal space $X = \mathbb{R}^{1,p,q}$. Suppose $g \in \mathrm{Spin}(X) = \{g \in \overset{+}{\Gamma}(X) : N(g) = 1 \text{ or } -1\}$; then $g = a + b\cdot\gamma^0$ with $a \in \overset{+}{C}(\{1,2,\cdots,n\})$, $b \in \overset{-}{C}(\{1,2,\cdots,n\})$; that is $a^\wedge = a$, $b^\wedge = -b$. Now $N(g) = \bar{g}g = (\bar{a} - \gamma^0\cdot\bar{b})(a + b\cdot\gamma^0)$, however $\gamma^0\bar{b} = -\bar{b}\gamma^0$ and $\gamma^0 a = a\gamma^0$, so $N(g) = \bar{a}a + (\bar{a}b + \bar{b}a)\cdot\gamma^0$. Thus $g \in \mathrm{Spin}(X)$ if and only if $g \in \overset{+}{\Gamma}(X)$ and $\bar{a}a = \pm 1$, $\bar{a}b + \bar{b}a = 0$.

$$\text{Define } H = \mathrm{Spin}(X) \cap \overset{+}{C}(\{1,2,\cdots,n\})$$

$$= \{a \in \overset{+}{C}(\{1,2,\cdots,n\}) : \bar{a}a = \pm 1\} \cap \overset{+}{\Gamma}(X)$$

and notice that $H \cong \mathrm{Spin}(p,q) < \mathrm{Spin}(1,p,q)$.

Also, define $K = \{1 + b\gamma^0 : b \in C(\{1,2,\cdots,n\})$, $\bar{b} = -b = b^\wedge\} \cap \overset{+}{\Gamma}(X)$.

We shall show that H,K satisfy the conditions of the lemma just referred to.

That H is a subgroup, is immediate from its definition. As for K, suppose $1+b\cdot\gamma^0 \in K$. Since $(1+b\cdot\gamma^0)^{-1} = 1-b\cdot\gamma^0$, the conditions $(1+b\cdot\gamma^0)^{-1}\gamma^i(1+b\cdot\gamma^0) = \gamma^i + (b\gamma^i+\gamma^i b)\cdot\gamma^0$ for $0 \leq i \leq n$ yield the condition $b\gamma^i+\gamma^i b \in R$, $0 \leq i \leq n$. For $i = 0$, we gain no new information as $b^\wedge = -b$ already implies this, but for $1 \leq i \leq n$, we find $b = \sum_{1\leq i\leq n} b_i\cdot\gamma^i \in R^{0,p,q} \subset X$ (this follows, for if $b = \sum_{|I|\text{odd}} b_I\gamma^I$, then $b\gamma^i+\gamma^i b \in R$ implies $\sum_{\substack{|I|\text{odd}\\ i\in I}} b_I\gamma^i\gamma^I = 0$, so $b_I = 0$ when $i \in I$, $|I|$ odd unless $I = \{i\}$; this being true for all i, $1 \leq i \leq n$, we have $b \in X$). Consequently, the condition that $1+b\cdot\gamma^0 \in \overset{+}{\Gamma}(X)$ already implies $b^\wedge = -b$, $b^- = -b$ and hence $K = \{1 + b\cdot\gamma^0 : b = \sum_{1\leq i\leq n} b_i\cdot\gamma^i\}$, and as such is clearly a subgroup of $Spin(X)$ isomorphic to R^n.

Next, we show that $Spin(X) = K\cdot H$. To this end, suppose $a+b\cdot\gamma^0 \in Spin(X)$, and notice that $(a+b\cdot\gamma^0)^{-1} = a^{-1}-a^{-1}b^{-1}a\cdot\gamma^0$. The condition that $a+b\cdot\gamma^0 \in \overset{+}{\Gamma}(X)$ now implies that $(a+b\cdot\gamma^0)^{-1}\gamma^i(a+b\cdot\gamma^0) = a^{-1}\gamma^i a + a^{-1}(\gamma^i\cdot ba^{-1}+ba^{-1}\cdot\gamma^i)a\cdot\gamma^0 \in X = R^{0,p,q} \oplus R^{1,0,0}$ and therefore that $a^{-1}\gamma^i a \in X$, $\gamma^i ba^{-1} + ba^{-1}\gamma^i \in R$. The first condition is merely that $a \in \overset{+}{\Gamma}(X)$ and hence $a \in H$, and the second condition that $ba^{-1} \in X$, hence $1 + ba^{-1}\cdot\gamma^0 \in K$. Thus we have $a+b\cdot\gamma^0 = (1+ba^{-1}\cdot\gamma^0)a \in K\cdot H$.

It is obvious that $H\cap K = \{1\}$. Furthermore, $K \triangleleft Spin(X)$ because if $1 + c\cdot\gamma^0 \in K$, $a+b\cdot\gamma^0 \in Spin(X)$ then $(a+b\cdot\gamma^0)^{-1}(1+c\cdot\gamma^0)(a+b\cdot\gamma^0) = 1 + a^{-1}ca\cdot\gamma^0 \in K$ (as $c \in X \Rightarrow a^{-1}ca \in X \Rightarrow 1 + a^{-1}ca\cdot\gamma^0 \in K$).

Finally, we compute $z(K) \cap H$. Let $a \varepsilon H$ commute with all members of K ; that is suppose $a(1+b\cdot\gamma^0) = (1+b\cdot\gamma^0)a$, $b \varepsilon R^{p,q} \subset X$. Then we must have $ab = ba$, $b \varepsilon R^{p,q}$ and knowing already that $a\gamma^0 = \gamma^0 a$ we have $ab = ba$, $b \varepsilon X$ so $a \varepsilon Z(R_{1,p,q})$. From Prop. I.1.7, $Z(R_{1,p,q}) \cap \overset{+}{R}_{1,p,q} = \overset{+}{R}_{1,0,0} = R$ and therefore $a \varepsilon R \Rightarrow$ $a \varepsilon \{-1,1\}$ since $\bar{a} a = a^2 \varepsilon \{-1,1\}$.

As the hypotheses of the required lemma are fulfilled, the proof is finished. □

From this theorem we obtain:

Corollary I.3.2.: $Spin(1,p,q) \cong R^n \text{Ⓢ} Spin(p,q)$, where $Spin(p,q)$ acts on R^n , $n = p+q \geq 1$, through the projection map $\rho : Spin(p,q) \to SO(p,q)$.

Proof: This is obvious from the structure of $Spin(1,p,q)$ as proved in Theorem I.3.1 (see also proof of Thm. I.3.4). □

Corollary I.3.3.: $Spin^+(1,p,q) \cong R^n \text{Ⓢ} Spin^+(p,q)$, and except when $(p,q) = (0,0),(0,1),(1,0),(1,1)$, $Spin^+(1,p,q) \overset{\rho}{\to} SO^+(1,p,q)$ is 2-1.

Proof: See the remarks in section 2 on $Spin^+(X)$ and use Corollary I.3.2 and the isomorphism $SO^+(1,p,q) \cong R^n \text{Ⓢ} SO^+(p,q)$. □

Theorem I.3.4.: $Pin(1,p,q)/\{-1,1\} \cong R^{n+1} \text{Ⓢ} (Pin(p,q)/\{-1,1\})$ where $n = p+q \geq 1$, and $Pin(p,q)/\{-1,1\} \cong O(p,q)$ acts on $R^{n+1} = R \oplus R^n$ as multiplication by $\det \rho$ on the first factor, and by the standard $O(p,q)$ linear action on the second factor.

Proof: We use the notation of Theorem I.3.1. Now $\mathrm{Pin}(X) = \{g \in \Gamma(X) : \bar{g}g = \pm 1\}$, so choose $g = a+b\cdot\gamma^0 \in \mathrm{Pin}(X)$ where $a,b \in C(\{1,2,\cdots,n\})$, and see what $\bar{g}g = \pm 1$ means. Since $\bar{g} = \bar{a}+\overline{\gamma^0}\bar{b} = \bar{a}-\gamma^0\bar{b} = \bar{a}-(\bar{b})^\wedge\gamma^0$ (recall that $\gamma^0\gamma^I = (-1)^{|I|}\gamma^I\gamma^0 = (\gamma^I)^\wedge\gamma^0$, so $\gamma^0\bar{b} = (\bar{b})^\wedge\gamma^0$) we have $\bar{g}g = (\bar{a}-(\bar{b})^\wedge\cdot\gamma^0)(a+b\cdot\gamma^0) = \bar{a}a + (\bar{a}b-(\bar{b}a)^\wedge)\cdot\gamma^0$, using the fact that $(\bar{b})^\wedge\gamma^0 a = (\bar{b})^\wedge a^\wedge\gamma^0 = (\bar{b}a)^\wedge\cdot\gamma^0$. Thus $\bar{g}g = \pm 1 \iff \bar{a}a = \pm 1$, $\bar{a}b = (\bar{b}a)^\wedge$, and therefore $\mathrm{Pin}(X) = \{a+b\cdot\gamma^0 \in \Gamma(X) : \bar{a}a = \pm 1,\ \bar{a}b = (\bar{b}a)^\wedge\}$.

In analogy with the proof of Theorem I.3.1 we define:

$$H = \mathrm{Pin}(X) \cap C(\{1,2,\cdots,n\})$$

$$= \{a \in C(\{1,2,\cdots,n\}) : \bar{a}a = \pm 1\} \cap \Gamma(X)$$

and remark that H is a subgroup of $\mathrm{Pin}(X)$ isomorphic to $\mathrm{Pin}(p,q)$.

Also, define $K = \{1+b\cdot\gamma^0 : b \in C(\{1,2,\cdots,n\}),\ b = (\bar{b})^\wedge\} \cap \Gamma(X)$. We now determine the structure of K as an (abelian) subgroup of $\mathrm{Pin}(X)$. As $(1+b\cdot\gamma^0)^{-1}\gamma^0(1+b\cdot\gamma^0)^\wedge = \gamma^0$, the restriction that $1+b\cdot\gamma^0 \in \Gamma(X)$ requires only that $(1+b\cdot\gamma^0)^{-1}\gamma^i(1+b\cdot\gamma^0) \in X$, $1 \le i \le n$. Computing and using the fact that $(1+b\cdot\gamma^0)^{-1} = 1-b\cdot\gamma^0$, we find that $\gamma^i b^\wedge - b\gamma^i \in R$, $1 \le i \le n$. If $b = \sum_I b_I\gamma^I$, then

$$\bar{b} = \sum_I b_I(-1)^{|I| + \frac{|I|(|I|-1)}{2}}\gamma^I,\quad b^\wedge = \sum_I b_I(-1)^{|I|}\gamma^I \quad \text{so} \quad \bar{b} = b^\wedge \Rightarrow$$

$b_I = 0$ when $|I| \equiv 2,3 \pmod 4$ (recall equation (1.4) and remarks following (1.5)). Writing out the expression for $\gamma^i b^\wedge - b\gamma^i$, we find that $\sum_{i\in I} b_I\gamma^I\gamma^i \in R$ with result that $b_I \ne 0$, $I \ne \phi \Rightarrow I = \{i\}$. Consequently, $b = b_\phi + \sum_{1\le i\le n} b_i\gamma^i$, $b_\phi, b_i \in R$ satisfying, a fortiori, the

condition $b = (b^{-})^{\wedge}$. Thus $K = \{1+(b_\phi + \sum_{1\leq i\leq n} b_i\gamma^i)\cdot\gamma^0\} \cong R^{n+1}$.

As expected, $\mathrm{Pin}(X) = K\cdot H$ (= $H\cdot K$ therefore). For let $a+b\cdot\gamma^0 \in \mathrm{Pin}(X)$ and require $a+b\cdot\gamma^0 \in \Gamma(X)$. This means that $(a+b\cdot\gamma^0)^{-1}\,\gamma^i(a+b\cdot\gamma^0)^{\wedge} = a^{-1}\gamma^i a^{\wedge} - a^{-1}(\gamma^i(b(a^{\wedge})^{-1})^{\wedge} - (b(a^{\wedge})^{-1})\gamma^i)a\cdot\gamma^0 \in X$ so $a^{-1}\gamma^i a^{\wedge} \in X$ and $\gamma^i(b(a^{\wedge})^{-1}) - (b(a^{\wedge})^{-1})\gamma^i \in R$. Thus $a \in H$, and $b(a^{\wedge})^{-1} \in R \oplus R^{0,p,q} \Rightarrow 1 + b(a^{\wedge})^{-1}\cdot\gamma^0 \in K$ with the result that $a+b\cdot\gamma^0 = (1+b(a^{\wedge})^{-1}\cdot\gamma^0)a \in K\cdot H$.

It is obvious that $H\cap K = \{1\}$.

To show that $K \lhd \mathrm{Pin}(X)$, we need the following classical result (Porteous (1969), Thm. 13.44) that $a \in \mathrm{Pin}(p,q)$ is representable as a product $a = a_1 a_2 \cdots a_m$, where each $a_i \in R^{p,q}$, and therefore $a^{\wedge} = (-1)^m a$. Suppose $a+b\cdot\gamma^0 \in \mathrm{Pin}(X)$, $1+c\cdot\gamma^0 \in K$, $c = c_\phi + \sum_{1\leq i\leq n} c_i\gamma^i$. Then $(a+b\cdot\gamma^0)(1+c\cdot\gamma^0)(a+b\cdot\gamma^0)^{-1} = 1 + ac(a^{\wedge})^{-1}\cdot\gamma^0$, using $(a+b\cdot\gamma^0)^{-1} = a^{-1} - a^{-1}b(a^{\wedge})^{-1}\cdot\gamma^0$. Since $c = c_\phi + \sum_{1\leq i\leq n} c_i\cdot\gamma^i \in R \oplus X$ and $aX(a^{\wedge})^{-1} = X$ we need only argue that $a(a^{\wedge})^{-1} \in R$. This, however, follows from the remarks above that $a^{\wedge} = \pm a$.

Finally, $z(K)\cap H = \{-1,1\}$. For if $a \in z(K)\cap H$, then $a(1+b\cdot\gamma^0) = (1+b\cdot\gamma^0)\,a \Rightarrow ab = ba^{\wedge}$, $b \in R \oplus R^{0,p,q}$. This requires $a \in R$ by Prop. I.1.5, and hence $a \in \{-1,1\}$.

As all the hypotheses of the previously quoted lemma are fulfilled, the stated isomorphism is proved by computing the action. Let $a+b\cdot\gamma^0$, $a'+b'\cdot\gamma^0 \in \mathrm{Pin}(X)$ and let's write $x_\phi + \vec{x}\cdot\vec{\gamma} = x_\phi + \sum_{1\leq i\leq n} x_i\gamma^i = b(a^{\wedge})^{-1}$ (since $1 + b(a^{\wedge})^{-1}\cdot\gamma^0 \in K$) and $x'_\phi + \vec{x}'\cdot\vec{\gamma} = b'(a'^{\wedge})^{-1}$.

Then, as $a+b\cdot\gamma^0 = (1+b(a^\wedge)^{-1}\cdot\gamma^0)a$ and $a'+b'\cdot\gamma^0 = (1+b'(a'^\wedge)^{-1}\cdot\gamma^0)a'$, we find that $(a+b\cdot\gamma^0)(a'+b'\cdot\gamma^0) = (1+(b(a^\wedge)^{-1} + a(b'(a'^\wedge)^{-1})(a^\wedge)^{-1})\cdot\gamma^0)aa'$.

Therefore, the mapping:

$$\tilde{\psi} : \mathrm{Pin}(1,p,q) \to \mathbb{R}^{n+1} \text{ Ⓢ } \mathrm{Pin}(p,q)$$

by $a+b\cdot\gamma^0 \to ((x_\phi,\underset{\to}{x}),a)$ with x_ϕ, $\underset{\to}{x}$ as defined as above, is an isomorphism. The multiplication law in $\mathbb{R}^{n+1}$ Ⓢ $\mathrm{Pin}(p,q)$ is as follows:

$$((x_\phi,\underset{\to}{x}),a)((x_\phi{}',\underset{\to}{x}'),a') = ((x_\phi+a(a^\wedge)^{-1}x_\phi{}',\underset{\to}{x}+\underset{\to}{x}'\cdot\rho(a^{-1})),aa') \quad .$$

Now because $a(a^\wedge)^{-1} = \pm 1$ according as $\rho(a)$ preserves/reverses orientations, $a(a^\wedge)^{-1} = \det \rho(a)$, and therefore we also have an isomorphism:

$$\psi : \mathrm{Pin}(1,p,q)/\{-1,1\} \to \mathbb{R}^{n+1} \text{ Ⓢ } O(p,q)$$

where multiplication in $\mathbb{R}^{n+1}$ Ⓢ $O(p,q)$ is as follows:

$$((x_\phi,\underset{\to}{x}),T)((x_\phi{}',\underset{\to}{x}'),T') = ((x_\phi+(\det T)\cdot x_\phi{}',\underset{\to}{x}+\underset{\to}{x}'\cdot T^{-1}),TT') \quad .$$

This establishes the theorem and elaborates further on Cor. I.3.2 and Cor. I.3.3 (by choosing $a \in \mathrm{Spin}(p,q)$, $x_\phi = 0$ etc.), and the following corollary. □

<u>Corollary I.3.5.</u>: $\mathrm{Pin}(1,p,q) \cong \mathbb{R}^{n+1}$ Ⓢ $\mathrm{Pin}(p,q)$, where $n = p+q$, and the left action of $\mathrm{Pin}(p,q)$ on $\mathbb{R}^{n+1}$ is as follows:

$$a\cdot(x_\phi,\underset{\to}{x}) = (\det \rho(a)\cdot x_\phi,\underset{\to}{x}\cdot\rho(a^{-1})) \quad .$$ □

Because of the importance of the physically interesting case $p=0$, $q=3$, we outline in detail the correspondence between $\mathrm{Spin}^+(1,0,3)$ and $SO^+(1,0,3)$: namely the two-to-one projection $\rho : \mathrm{Spin}^+(1,0,3) \to SO^+(1,0,3)$.

In keeping with previous notation, $\{\gamma^i\}$, $0 \leq i \leq 3$, will denote an orthonormal basis of $R^{1,0,3}$ which generates $R_{1,0,3}$. The Galilei group $G_6 = R^3 \circledS SO(3)$ acts on a basis frame (e_i) , $0 \leq i \leq 3$, of R^4 as follows (implied summation over repeated raised, lowered indices):

$$(e_i,\Lambda) \to \hat{e}_i = e_j(\Lambda^{-1})^j{}_i \tag{3.5a}$$

where as outlined earlier in this section, $\Lambda\gamma\Lambda^t = \gamma$ and $\det \Lambda = 1$, hence $\Lambda^0{}_0 = 1$, $\Lambda^0{}_B = 0$, $\Lambda^A{}_0 = v^A$, $\Lambda^A{}_B = R^A{}_B$, $1 \leq A,B \leq 3$. This action on frames induces an action on coordinates:

$$x^i \to \hat{x}^i = \Lambda^i{}_j\, x^j \tag{3.5b}$$

in such a way that, $x^i e_i = \hat{x}^i \hat{e}_i$ is invariant.

The dual action on dual coframes (θ^i) (i.e. frames in the dual space satisfying $\theta^i \cdot e_j = \delta^i{}_j$ $0 \leq i , j \leq 3$) is:

$$(\theta^i,\Lambda) \to \hat{\theta}^i = \Lambda^i{}_j\, \theta^j \tag{3.6a}$$

and on covariant components x_i of a vector:

$$x_i \to \hat{x}_i = x_j(\Lambda^{-1})^j{}_i \quad . \tag{3.6b}$$

Our notation γ^i (as opposed to γ_i) anticipates the interpretation of γ^i as a member of an orthonormal coframe, and consequently the covering map $\rho : \mathrm{Spin}(1,0,3) = \mathrm{Spin}^+(1,0,3) \to SO^+(1,0,3) \cong R^3 \circledS SO(3) = G_6$ defined, as before, by $\gamma^i \to \gamma^i \cdot \rho(g) = g^{-1} \gamma^i g^\wedge = g^{-1} \gamma^i g$, determines $\Lambda(g) \in G_6$, where $\Lambda(g)^i{}_j \gamma^j = g^{-1} \gamma^i g^\wedge$. In this way one may identify θ^i , γ^i for they both transform in the same manner (2.6a).

Corollary I.3.6.: From Thm. I.3.1 we have:

$$\mathrm{Spin}(1,0,3) = \Big\{ a+b\cdot\gamma^0\gamma^1\gamma^2\gamma^3 \;:\; a=a^0+a^1\gamma^2\gamma^3+a^2\gamma^3\gamma^1+a^3\gamma^1\gamma^2 ,$$
$$b = b^0+b^1\gamma^2\gamma^3+b^2\gamma^3\gamma^1+b^3\gamma^1\gamma^2 , (a^0)^2+(a^1)^2+(a^2)^2+(a^3)^2 = 1 ,$$
$$a^0b^0+a^1b^1+a^2b^2+a^3b^3 = 0 \Big\} .$$

Proof: Simply modify the notation of the proof of Thm. I.3.1 as follows: $a_\phi \to a^0$, $a_{23} \to a^1$, $a_{13} \to -a^2$, $a_{12} \to a^3$, $b_1 \to -b^1$, $b_2 \to -b^2$, $b_3 \to -b^3$, $b_{123} \to -b^0$ and notice that $1 = \bar{a} a = (a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$ and $\bar{a}(-b\cdot\gamma^1\gamma^2\gamma^3)+(-b\cdot\gamma^1\gamma^2\gamma^3)^- a = 0 \iff a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3 = 0$. That $a+b\cdot\gamma^0\gamma^1\gamma^2\gamma^3 \in \overset{+}{\Gamma}(R^{1,0,3})$, poses no additional conditions. □

Given $(a^0,\vec{a}) = (a^0,a^1,a^2,a^3) \in S^3$, the unit Euclidean 3-sphere in R^4 , and $(b^0,\vec{b}) = (b^0,b^1,b^2,b^3) \in T_{(a^0,\vec{a})} S^3$, the tangent space to S^3 at $(a^0,\vec{a})$ (tangency requires that $a^0b^0 + \vec{a}\cdot\vec{b} = a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3 = 0$) , let us define:

$$R = ((a^0)^2-\vec{a}\cdot\vec{a})1_3 + 2\,\vec{a}\cdot\vec{a}\; P_{\vec{a}} + 2a^0\, J_{\vec{a}} \tag{3.7a}$$

$$\vec{v} = 2(-a^0\vec{b}+b^0\vec{a}-\vec{a}\times\vec{b}) \tag{3.7b}$$

where 1_3 is the identity map of $\mathbb{R}^3$, $P_{\vec{a}}$ is the orthogonal projection parallel to $\vec{a}$ ($P_{\vec{a}}\,\vec{x} = (\vec{a}\cdot\vec{a})^{-1}(\vec{a}\cdot\vec{x})\vec{a}$), and $J_{\vec{a}}\,\vec{x} = \vec{a}\times\vec{x}$ (cross product of vectors in $\mathbb{R}^3$). Then, $(\vec{v},R) \in G_6$ and with $g = a+b\cdot\gamma^0\gamma^1\gamma^2\gamma^3$:

$$\hat{\gamma}^0 = g^{-1}\gamma^0 g = \gamma^0 \quad , \quad \hat{\gamma}^A = g^{-1}\gamma^A g = v^A\gamma^0 + R^A{}_B\,\gamma^B \tag{3.8}$$

where $1 \leq A,B \leq 3$.

Conversely, if $(\vec{v},R) \in G_6$, define $\vec{a} \in \mathbb{R}^3$ up to $\pm$ sign, and $a^0 \in \mathbb{R}$ by:

$$R\vec{a} = \vec{a} \ , \ \vec{a}\cdot\vec{a} = \sin^2(\tfrac{1}{2}\,\omega) \ , \ a^0 = \cos(\tfrac{1}{2}\,\omega) \tag{3.9a}$$

where ω = angle of right-handed rotation about $\vec{a}$ defined by R . Also define:

$$b^0 = \tfrac{1}{2}(\vec{a}\cdot\vec{v}) \quad , \quad \vec{b} = -\tfrac{1}{2}(a^0\vec{v} - \vec{a}\times\vec{v}) \tag{3.9b}$$

and note then, that $a+b\cdot\gamma^0\gamma^1\gamma^2\gamma^3 \in \mathrm{Spin}(1,0,3)$. The two-to-one ambiguity arises out of the choice of $\vec{a}$: choosing $-\vec{a}$ instead, causes the replacements $\omega \to 2\pi-\omega$, $a^0 \to -a^0$, $b^0 \to -b^0$, $\vec{b} \to -\vec{b}$ resulting in $-(a+b\cdot\gamma^0\gamma^1\gamma^2\gamma^3) \in \mathrm{Spin}(1,0,3)$.

Before closing this section, there are several aspects of the results on pin and spin groups deserving of comment.

First of all, the mapping $\rho : \mathrm{Pin}(1,p,q) \to O(1,p,q)$ by $x \to x\cdot\rho(g) = g^{-1}xg^{\wedge}$, $x \in X$, is <u>not</u> onto. In fact it has a non-trivial kernel $\ker\rho = \{\pm\,(1+b_\phi\cdot\gamma^0) : b_\phi \in \mathbb{R}\}$ (for, suppose $\rho(g) = 1_X$; then $g^{-1}xg^{\wedge} = x$, $x \in X$ implies, by Prop. I.1.5, $g \in \mathbb{R}_{1,0,0}$ and consequently

$g = a+b_{\phi}\cdot\gamma^0$ with $a^2 = N(g) = \pm 1$). Actually, from the fact that:

$$0(1,p,q) = \left\{ \begin{pmatrix} \alpha & 0 \\ \vec{\beta} & L \end{pmatrix} : 0 \neq \alpha \ , \ L \in 0(p,q) \ , \ \vec{\beta} \in \mathsf{R}^n \right\}$$

one easily shows that $0(1,p,q) \cong (\mathsf{R}^n \circledS 0(p,q)) \circledS \overset{*}{\mathsf{R}}$, where $\overset{*}{\mathsf{R}} = \mathsf{R}\backslash\{0\}$, and $\overset{*}{\mathsf{R}}$ acts on $\mathsf{R}^n \circledS 0(p,q)$ by $\alpha\cdot(\vec{\beta},L) = (\vec{\beta}\,\alpha^{-1},L)$. It is then clear why $\rho(\mathrm{Pin}(1,p,q)) \neq 0(1,p,q)$; it happens that $\mathrm{Pin}(1,p,q)$ and $0(1,p,q)$ are not even locally isomorphic as Lie groups (of course non-surjectivity of ρ is expected because degeneracy of $X \Rightarrow 0(X)$ is non-semisimple => orthogonal automorphisms need not be inner i.e. of the form $\rho(g)$).

Secondly, non-surjectivity of ρ as a mapping from spin groups to special orthogonal groups becomes an issue when the orthogonal space is of the type (r,p,q) with $r \geq 2$. We shall return briefly to this point in section 5.

Finally, making the choice $p = 0$, $q = 3$ in Thm. I.3.1, Cor. I.3.2, Cor. I.3.4, we recover the Galilei situation and remark that everything turns out as nicely as could have been expected.

I.4. LIE ALGEBRAS

If (X,B) denotes the orthogonal space $\mathbb{R}^{r,p,q}$, then the orthogonal and special orthogonal groups $O(X,B)$ $(= O(B))$ and $SO(X,B)$ $(= SO(B))$, also denoted by notational abuse $O(X)$, $SO(X)$ are usually defined thus:

$$O(B) = \{\Lambda \in GL(X) : B(x\Lambda, y\Lambda) = B(x,y) \ , \ x,y \in X\}$$

$$= \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} : a \in GL(r) \ , \ d \in O(p,q) \ , \ b \in \mathrm{End}(\mathbb{R}^{p+q}, \mathbb{R}^r) \right\} \tag{4.1}$$

$$SO(B) = \{\Lambda \in O(B) : \det \Lambda = 1\}$$

$$= \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \in O(B) : \det a \cdot \det d = 1 \right\} \tag{4.2}$$

In addition: $SO^+(B) = \{\Lambda \in SO(B) : \Lambda$ preserves orientation on each component of the decomposition $\mathbb{R}^{r,p,q} = \mathbb{R}^{r,0,0} \oplus \mathbb{R}^{0,p,0} \oplus \mathbb{R}^{0,0,q}\}$

$$= \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} : a \in SL(r) \ , \ d \in SO^+(p,q) \right\} \tag{4.3}$$

= connected component containing the identity.

These groups are all Lie groups (being closed subgroups of general linear groups) and therefore possess Lie algebras. Ordinary matrix exponentiation is the exponential map from the Lie algebra to the group, and using this fact we find for the Lie algebras:

$$o(B) = \{\lambda \in \mathrm{End}(X) : B(x\lambda, y) + B(x, y\lambda) = 0 \ , \ x,y \in X\}$$

$$= \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} : a \in \mathrm{End}(\mathbb{R}^r) \ , \ d \in o(p,q) \ , \ b \in \mathrm{End}(\mathbb{R}^{p+q}, \mathbb{R}^r) \right\} \tag{4.4}$$

$$\mathfrak{so}(B) = \{\lambda \in \mathfrak{o}(B) : \mathrm{tr}(\lambda) = 0\} \qquad \mathrm{tr} = \text{trace}$$
$$= \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \in \mathfrak{o}(B) : \mathrm{tr}(a) = 0 = \mathrm{tr}(d) \right\} \tag{4.5}$$

$$\mathfrak{so}^{+}(B) = \mathfrak{so}(B) \tag{4.6}$$

For the present, we are only interested in the cases $r = 0,1$; $r \geq 2$ will be touched on in section 5. Only $\mathrm{Spin}^{+}(r,p,q)$ and $\mathrm{SO}^{+}(r,p,q)$ will be considered in the Lie algebra context (see Notes).

The Lie group homomorphism for $r \leq 1$ (see Cor. I.3.3) :

$$\rho : \mathrm{Spin}^{+}(r,p,q) \to \mathrm{SO}^{+}(r,p,q) \tag{4.7}$$

has kernel $\{-1,1\}$, therefore is a 2-1 covering map (for the relation between $\mathrm{Spin}^{+}(r,p,q)$ and the universal covering group of $\mathrm{SO}^{+}(r,p,q)$ when $r \leq 1$, refer to the Notes). The tangent (or derivative) mapping ρ_* is then an isomorphism at each point of $\mathrm{Spin}^{+}(r,p,q)$ and therefore, a fortiori, a Lie algebra isomorphism (conversely, Lie groups with isomorphic Lie algebras are locally isomorphic):

$$\rho_* : \mathfrak{spin}(r,p,q) \to \mathfrak{so}(r,p,q) \tag{4.8}$$

We now make the mapping ρ_* explicit. As usual, $\{\gamma^i\}$ will form an orthonormal basis of $R^{r,p,q}$ generating $R_{r,p,q}$; A and $\mathfrak{s}$ will denote members of $\mathfrak{so}(r,p,q)$ and $\mathfrak{spin}(r,p,q)$ respectively; there will be summation implied over repeated raised and lowered indices.

One has:

$$A \in so(r,p,q) \iff (e^{tA})^i{}_k\, B^{k\ell}\, (e^{tA})^j{}_\ell = B^{ij} \quad , \quad t \in \mathbb{R}$$

$$\iff A^i{}_k B^{k\ell} \delta^j{}_\ell + \delta^i{}_k B^{k\ell} A^j{}_\ell = 0$$

$$\iff A^i{}_k B^{kj} + B^{ik} A^j{}_k = 0 \tag{4.9}$$

Now

$$s = \rho_*^{-1}(A) \iff \gamma^i \cdot \rho(e^{ts}) = (e^{tA})^i{}_j\, \gamma^j \quad , \quad t \in \mathbb{R}$$

$$\iff e^{-ts}\, \gamma^i\, e^{ts} = (e^{tA})^i{}_j\, \gamma^j \quad , \quad t \in \mathbb{R}$$

$$\iff [\gamma^i, s] = A^i{}_j\, \gamma^j \tag{4.10}$$

Solutions s , s' of (4.10) differ by a real number (for if s , s' solve (4.10), then $[\gamma^i, s'-s] = 0$ and $s'-s \in Z(\mathbb{R}_{r,p,q})$; however $s \in spin(r,p,q) \Rightarrow e^{ts} \in \overset{+}{C} \Rightarrow s \in \overset{+}{C}$, hence Prop. I.1.7 $\Rightarrow$ $s'-s \in Z(\mathbb{R}_{r,p,q}) \cap \overset{+}{\mathbb{R}}_{r,p,q} = \overset{+}{\mathbb{R}}_{r,0,0} = \mathbb{R}$ for $r = 0,1$). In fact, the solution is unique (if $s' = a+s$ for some $a \in \mathbb{R}$, then $\forall t \in \mathbb{R}$, $e^{ts'} = e^{ta} \cdot e^{ts}$ and $(e^{ts'})^- = e^{ta} \cdot (e^{ts})^- = e^{ta} \cdot e^{ts^-}$; but we also have $(e^{ts})^-(e^{ts}) = \pm 1$ and likewise for $e^{ts'}$, and therefore find that $e^{2ta} = \pm 1 \Rightarrow a = 0$ and $s' = s$).

The solution to (4.10) is:

$$s = \frac{1}{4} A_{ij}\, \gamma^i \gamma^j \tag{4.11}$$

where $A^i{}_j = -B^{ik} A_{kj}$ and $A_{ij} + A_{ji} = 0$. In fact, trying a solution $s = \frac{1}{4} A_{k\ell}\, \gamma^k \gamma^\ell$ in (4.10) requires (assuming $A_{k\ell} = -A_{\ell k}$) $A^i{}_j = -B^{ik} A_{kj}$ which because $r \leq 1$ has a unique solution A_{ij} , anti-symmetric in i,j . Explicitly:

$$A_{KL} = -B_{KM}\, A^M{}_L \quad \text{if} \quad r = 0 \quad \text{and} \quad 1 \leq K,L,M \leq n$$

and $B_{KL} = B^{KL}$ are the entries of B^{-1} .

$A_{KL} = -B_{KM}\, A^M{}_L$, $A_{0L} = B_{LM}\, A^M{}_0$ if $r = 1$ and $1 \leq K,L,M \leq n-1$ with $B_{KL} = B^{KL}$ the entries of the inverse of the non-singular part of B .

It becomes apparent then, that (for $r \leq 1$):

$$\mathit{spin}(r,p,q) = \operatorname{span}_{\mathbb{R}}\{\gamma^i\gamma^j : i < j\}$$

where the Lie product $[\ ,\ ]$ is the commutator. This of course may be verified directly, as $\operatorname{span}_{\mathbb{R}}\{\gamma^i\gamma^j : i < j\}$ is closed under $[\ ,\ ]$ and in fact $\{\frac{1}{2}\gamma^i\gamma^j : i < j\}$ forms a basis of $\mathit{spin}(r,p,q)$ with the usual structure constants ($\frac{1}{2}\gamma^i\gamma^j$ corresponds to an infinitesimal "rotation" in the plane (i,j)): $[\frac{1}{2}\gamma^i\gamma^j , \frac{1}{2}\gamma^k\gamma^\ell] = B^{ik}\cdot\frac{1}{2}\gamma^j\gamma^\ell - B^{jk}\cdot\frac{1}{2}\gamma^i\gamma^\ell + B^{i\ell}\cdot\frac{1}{2}\gamma^k\gamma^j - B^{j\ell}\cdot\frac{1}{2}\gamma^k\gamma^i$.

We close this section by displaying explicit (physically motivated) bases of the Lie algebras $\mathit{spin}(1,4)$, $\mathit{spin}(1,3)$, $\mathit{spin}(1,0,3)$. As remarked earlier, the orthogonal inclusions $\mathbb{R}^{1,3} \subset \mathbb{R}^{1,4}$ $\mathbb{R}^{1,0,3} \subset \mathbb{R}^{1,4}$ imply the Clifford inclusions $\mathbb{R}_{1,3} \subset \mathbb{R}_{1,4}$, $\mathbb{R}_{1,0,3} \subset \mathbb{R}_{1,4}$ and hence the Lie inclusions $\mathit{spin}(1,3) \subset \mathit{spin}(1,4)$, $\mathit{spin}(1,0,3) \subset \mathit{spin}(1,4)$.

To begin, let γ^i , $0 \leq i \leq 4$, be an orthonormal basis of $\mathbb{R}^{1,4}$ generating $\mathbb{R}_{1,4}$.

Define:

$$\underset{S}{J}^A = \frac{1}{4}\,\epsilon^A{}_{BC}\,\gamma^B\gamma^C \quad , \quad \underset{S}{K}^A = \frac{1}{2}\,\gamma^0\gamma^A$$

$$\underset{S}{P}^A = \frac{1}{2}\,\gamma^A\gamma^4 \quad , \quad \underset{S}{H} = \frac{1}{2}\,\gamma^0\gamma^4 \tag{4.12}$$

where $1 \leq A,B,C \leq 3$; $\epsilon^A{}_{BC}$ = sign of permutation (ABC) of (123) and vanishes if A,B,C non-distinct.

The $\underset{S}{J}^A$, $\underset{S}{K}^A$, $\underset{S}{P}^A$, $\underset{S}{H}$ form a basis of the de Sitter Lie algebra $\mathit{spin}\,(1,4)$:

$$[\underset{S}{J}^A,\underset{S}{J}^B] = \epsilon^{AB}{}_C\underset{S}{J}^C \ , \ [\underset{S}{J}^A,\underset{S}{K}^B] = \epsilon^{AB}{}_C\underset{S}{K}^C \ , \ [\underset{S}{J}^A,\underset{S}{P}^B] = \epsilon^{AB}{}_C\underset{S}{P}^C \tag{4.13a}$$

$$[\underset{S}{K}^A,\underset{S}{K}^B] = -\epsilon^{AB}{}_C\underset{S}{J}^C \ , \ [\underset{S}{K}^A,\underset{S}{P}^B] = -\delta^{AB}\underset{S}{H} \ , \ [\underset{S}{P}^A,\underset{S}{P}^B] = \epsilon^{AB}{}_C\underset{S}{J}^C \tag{4.13b}$$

$$[\underset{S}{J}^A,\underset{S}{H}] = 0 \ , \ [\underset{S}{K}^A,\underset{S}{H}] = -\underset{S}{P}^A \ , \ [\underset{S}{P}^A,\underset{S}{H}] = -\underset{S}{K}^A \tag{4.13c}$$

The Lorentz Lie algebra $\mathit{spin}(1,3)$, obtained by the embedding $i : R_{1,3} \to R_{1,4}$ sending $\gamma^0,\gamma^1,\gamma^2,\gamma^3$ to themselves, induces the Lie algebra inclusion:

$$\underset{L}{J}^A = \underset{S}{J}^A \quad , \quad \underset{L}{K}^A = \underset{S}{K}^A \tag{4.14}$$

with Lie products as in (4.13).

The Galilei Clifford inclusion $R_{1,0,3} \subset R_{1,4}$, as mentioned in section 1, induces the Lie inclusion:

$$\underset{G}{J}^A = \underset{S}{J}^A \quad , \quad \underset{G}{K}^A = \frac{1}{2}\,(\gamma^0+\gamma^4)\gamma^A \tag{4.15}$$

and consequently the products:

$$[\underset{G}{J}^A,\underset{G}{J}^B] = \epsilon^{AB}{}_C\underset{G}{J}^C \ , \ [\underset{G}{J}^A,\underset{G}{K}^B] = \epsilon^{AB}{}_C\underset{G}{K}^C \ , \ [\underset{G}{K}^A,\underset{G}{K}^B] = 0 \quad . \tag{4.16}$$

I.5. HIGHER DEGENERATE SPACES: SPIN(2,0,3)

The theorems and corollaries of section 3, in large part, concern spin and pin groups of type $(1,p,q)$, and it was found, roughly speaking, that $\mathrm{Spin}(1,p,q)$ is the two-fold covering group of $SO(1,p,q)$ which is, again roughly, the inhomogeneous $SO(p,q)$ i.e., the semi-direct product of $\mathbb{R}^{p+q}$ with $SO(p,q)$. Put another way, the obvious spin group analogue of $\mathbb{R}^{p+q} \circledS SO(p,q)$ is surely $\mathbb{R}^{p+q} \circledS \mathrm{Spin}(p,q)$ (the action on $\mathbb{R}^{p+q}$ being through ρ) and it is no small anticlimax that this is exactly what the generalized definition yields! The details work out without wrinkles, almost too easily. Things, however, become more interesting for spin groups of type $(2,p,q)$.

Leaving this aside momentarily, the transformation group which incorporates the homogeneity of non-relativistic space-time is the *Galilei group* $G_{10} \cong \mathbb{R}^4 \circledS G_6$, where G_6 acts on $\mathbb{R}^4$ in the standard linear manner. G_{10} is the non-relativistic analogue of the *Poincaré* (= *inhomogeneous Lorentz*) *group* $L_{10} \cong \mathbb{R}^4 \circledS L_6$ of relativistic physics. Quantum mechanically, one is naturally led to consider the corresponding spin groups $\mathbb{R}^4 \circledS \mathrm{Spin}(1,0,3)$ and $\mathbb{R}^4 \circledS \mathrm{Spin}(1,3)$. It is in the interpretation of quantum theory that the differences between non-relativistic and relativistic kinematics become great. Whereas elementary particles in relativistic quantum theory are defined with reference to certain representations of $\mathbb{R}^4 \circledS \mathrm{Spin}(1,3)$, such is not, and indeed cannot be the case for $\mathbb{R}^4 \circledS \mathrm{Spin}(1,0,3)$ in the non-relativistic theory (Inönü, Wigner (1952)). It is the *extended Galilei group* G_{11} of dimension 11, which is required; that is to say, its spin analogue. The appearance of G_{11} rather than the more intuitive G_{10} is typically seen a posteriori,

following the quantum mechanical formalism, but rarely is it justified in a purely group theoretical manner.

In seeking such a natural development, one cannot help but notice these facts:

(i) the groups $R^{p+q} \circledS SO(p,q)$, $p+q \geq 3$, have trivial local central extensions (Bargmann (1954)) e.g. L_{10} has only trivial such extensions

(ii) $R^4 \circledS (R^3 \circledS SO(3))$ has a non-trivial local central extension

(iii) the semi-direct product structure of $Spin(1,p,q)$ depends crucially on the degeneracy of the space $R^{1,p,q}$, in particular on the appearance of a single orthonormal basis member γ^0 , such that $(\gamma^0)^2 = 0$.

One wonders whether another such basis element with zero square, when incorporated into the Clifford algebra framework could account for the double semi-direct product structure of G_{10} . In particular, what is $Spin(2,0,3)$?

Not wanting to dwell too heavily on G_{11} or extensions generally, we concentrate here on the groups $Spin(2,p,q)$, which do however bear some relation to extended inhomogeneous orthogonal groups.

<u>Theorem I.5.1.</u>: $Spin(2,p,q) \cong K(p,q) \circledS Spin(p,q)$, $Spin(2,p,q)\backslash\{-1,1\} \cong K(p,q) \circledS (Spin(p,q)\backslash\{-1,1\})$, where $K(p,q)$ is a non-trivial, one dimensional, central extension of the abelian group $R^n \times R^n$, $n = p+q \geq 1$, isomorphic to a Heisenberg group of dimension $2n+1$.

Proof: The strategy used in the proofs of Thm. I.3.1, Thm. I.3.4 is repeated, especially the lemma of section 3. We let $\gamma^0_1, \gamma^0_2, \gamma^1, \gamma^2, \cdots, \gamma^n$, $n = p+q$, denote an orthonormal basis of $X = \mathrm{R}^{2,p,q}$ which generates $\mathrm{R}_{2,p,q}$, with $(\gamma^0_1)^2 = 0 = (\gamma^0_2)^2$.

Let $g = a + b_1\cdot\gamma^0_1 + b_2\cdot\gamma^0_2 + b_{12}\cdot\gamma^0_1\gamma^0_2 \in \mathrm{Spin}(2,p,q)$, with $a^\wedge = a$, $b_1^\wedge = -b_1$, $b_2^\wedge = -b_2$, $b_{12}^\wedge = b_{12}$ and $a, b_1, b_2, b_{12} \in C(\{1,2,\cdots,n\})$. We examine the restriction $g^- g = \pm 1$; $g^- = a^- + b_1^-\cdot\gamma^0_1 + b_2^-\cdot\gamma^0_2 - b_{12}^-\cdot\gamma^0_1\gamma^0_2$ since $\gamma^0_1\gamma^I = (-1)^{|I|}\gamma^I\gamma^0_1$, $\gamma^0_2\gamma^I = (-1)^{|I|}\gamma^I\gamma^0_2$. Thus $g^-g = a^-a + (a^-b_1+b_1^-a)\cdot\gamma^0_1 + (a^-b_2+b_2^-a)\cdot\gamma^0_2 + (a^-b_{12} - b_{12}^-a - b_1^-b_2 + b_2^-b_1)\gamma^0_1\gamma^0_2$ and $g^-g = \pm 1 \iff a^-a = \pm 1$, $a^-b_1 + b_1^-a = 0$, $a^-b_2 + b_2^-a = 0$, $a^-b_{12} - b_{12}^-a - b_1^-b_2 + b_2^-b_1 = 0$. Equivalently, $g^-g = \pm 1$ if and only if $a^-a = \pm 1$, $(b_1a^{-1})^- = -b_1a^{-1}$, $(b_2a^{-1})^- = -b_2a^{-1}$, $b_{12}a^{-1} - (b_{12}a^{-1})^- = (b_1a^{-1})^-(b_2a^{-1}) - (b_2a^{-1})^-(b_1a^{-1}) = -[b_1a^{-1}, b_2a^{-1}]$.

We now require $g \in \Gamma(X)$. There are no restrictions imposed by demanding $g^{-1}\gamma^0_1 g \in X$, $g^{-1}\gamma^0_2 g \in X$; in fact, $g^{-1}\gamma^0_1 g = \gamma^0_1$, $g^{-1}\gamma^0_2 g = \gamma^0_2$, where $g^{-1} = a^{-1} + (-a^{-1}b_1a^{-1})\cdot\gamma^0_1 + (-a^{-1}b_2a^{-1})\cdot\gamma^0_2 + (-a^{-1}b_{12}a^{-1} - a^{-1}b_1a^{-1}b_2a^{-1} + a^{-1}b_2a^{-1}b_1a^{-1})\cdot\gamma^0_1\gamma^0_2$. Now, for $1 \le i \le n$:
$g^{-1}\gamma^i g = a^{-1}\gamma^i a + a^{-1}(\gamma^i\cdot b_1a^{-1} + b_1a^{-1}\cdot\gamma^i)a\cdot\gamma^0_1 + a^{-1}(\gamma^i\cdot b_2a^{-1} + b_2a^{-1}\cdot\gamma^i)a\cdot\gamma^0_2 + a^{-1}(\gamma^i\cdot b_{12}a^{-1} - b_{12}a^{-1}\cdot\gamma^i - b_1a^{-1}(\gamma^i\cdot b_2a^{-1} + b_2a^{-1}\cdot\gamma^i) + b_2a^{-1}(\gamma^i\cdot b_1a^{-1} + b_1a^{-1}\cdot\gamma^i))a\cdot\gamma^0_1\gamma^0_2$
so $g^{-1}\gamma^i g \in X \iff a^{-1}\gamma^i a \in \mathrm{R}^{0,p,q}$, $\gamma^i\cdot b_1a^{-1} + b_1a^{-1}\cdot\gamma^i \in \mathrm{R}$, $\gamma^i\cdot b_2a^{-1} + b_2a^{-1}\cdot\gamma^i \in \mathrm{R}$, $[\gamma^i, b_{12}a^{-1}] = b_1a^{-1}(\gamma^i\cdot b_2a^{-1} + b_2a^{-1}\cdot\gamma^i) - b_2a^{-1}(\gamma^i\cdot b_1a^{-1} + b_1a^{-1}\cdot\gamma^i)$. Since $a^-a = \pm 1$ and $a^{-1}\gamma^i a \in \mathrm{R}^{0,p,q}$, $1 \le i \le n$, we have $a \in H = \mathrm{Spin}(2,p,q) \cap \overset{+}{C}(\{1,2,\cdots,n\}) \cong \mathrm{Spin}(p,q)$. Following arguments used in the proof of Thm. I.3.1, $\gamma^i\cdot b_1a^{-1} + b_1a^{-1}\cdot\gamma^i \in \mathrm{R}$ and $\gamma^i\cdot b_2a^{-1} + b_2a^{-1}\cdot\gamma^i \in \mathrm{R} \iff b_1a^{-1}, b_2a^{-1} \in \mathrm{R}^{0,p,q}$; say,

$\underset{1}{ba}^{-1} = \sum_{1\leq i\leq n} \underset{1}{x}_i\gamma^i = \underset{\vec{1}}{x}\cdot\vec{\gamma}$, $\underset{2}{ba}^{-1} = \sum_{1\leq i\leq n} \underset{2}{x}_i\gamma^i = \underset{\vec{2}}{x}\cdot\vec{\gamma}$, as in Theorem I.3.4.

Then $\gamma^i\cdot\underset{1}{ba}^{-1} + \underset{1}{ba}^{-1}\cdot\gamma^i = -2B^{ii}\,\underset{1}{x}_i$ and $\gamma^i\cdot\underset{2}{ba}^{-1} + \underset{2}{ba}^{-1}\cdot\gamma^i = -2B^{ii}\,\underset{2}{x}_i$ (no summation implied). Incidentally, the conditions $(\underset{1}{ba}^{-1})^- = -\underset{1}{ba}^{-1}$, $(\underset{2}{ba}^{-1})^- = -\underset{2}{ba}^{-1}$ are now redundant. Finally, there remain only:

$$\underset{12}{ba}^{-1} - (\underset{12}{ba}^{-1})^- = -[\underset{1}{ba}^{-1},\underset{2}{ba}^{-1}] = \sum_{1\leq i<j\leq n} (-2)(\underset{1}{x}_i\underset{2}{x}_j-\underset{2}{x}_i\underset{1}{x}_j)\cdot\gamma^i\gamma^j$$

and

$$[\gamma^i, \underset{12}{ba}^{-1}] = \underset{1}{ba}^{-1}(-\ B^{ii}\underset{2}{x}_i) - \underset{2}{ba}^{-1}(-2B^{ii}\underset{1}{x}_i)$$

$$= \sum_{1\leq j\leq n} 2B^{ii}(\underset{1}{x}_i\underset{2}{x}_j-\underset{2}{x}_i\underset{1}{x}_j)\cdot\gamma^j \quad .$$

If $\underset{12}{ba}^{-1} = \sum_{|I|\,\text{even}} x_I\cdot\gamma^I$, then $\underset{12}{ba}^{-1} - (\underset{12}{ba}^{-1})^- = \sum_{|I|\equiv 2(\text{mod }4)} 2x_I\cdot\gamma^I$ (since $|I| \equiv 0(\text{mod }4) \Rightarrow (\gamma^I)^- = \gamma^I$) and consequently $x_{ij} = -(\underset{1}{x}_i\underset{2}{x}_j-\underset{2}{x}_i\underset{1}{x}_j)$, $i < j$, and $x_I = 0$ if $|I| \equiv 2(\text{mod }4)$ and $|I| > 2$. This leaves only x_I with $|I| \equiv 0$ (mod 4) to be determined. Now $[\gamma^i, \underset{12}{ba}^{-1}] = \sum_{\substack{|I|\,\text{even}\\ i\in I}} 2x_I\cdot\gamma^i\gamma^I$, but we know that apart from $|I| = 2$, the only possible non-empty surviving x_I 's are those with $|I| \equiv 0(\text{mod }4)$. However, $I \neq \phi$, $|I| \equiv 0(\text{mod }4)$, $i \in I \Rightarrow x_I = 0$ since $[\gamma^i, \underset{12}{ba}^{-1}] \in X$. Finally then, we have:

$$\underset{12}{ba}^{-1} = x_\phi - \sum_{1\leq i<j\leq n} (\underset{1}{x}_i\underset{2}{x}_j-\underset{2}{x}_i\underset{1}{x}_j)\cdot\gamma^i\gamma^j$$

$$= x_\phi - \frac{1}{2}\,[\underset{1}{ba}^{-1},\underset{2}{ba}^{-1}]$$

with x_ϕ arbitrary. Note that $g = (1+\underset{1}{ba}^{-1}\cdot\underset{1}{\gamma}^0+\underset{2}{ba}^{-1}\cdot\underset{2}{\gamma}^0+\underset{12}{ba}^{-1}\cdot\underset{1}{\gamma}^0\underset{2}{\gamma}^0)\cdot a$.

To summarize thus far: $\mathrm{Spin}(2,p,q) = K\cdot H$, where $H = \mathrm{Spin}(X) \cap \overset{+}{C}(\{1,2,\cdots,n\}) \cong \mathrm{Spin}(p,q)$, and $K = \left\{ 1 + (\vec{x}_1\cdot\vec{\gamma})\cdot\gamma^0_1 + (\vec{x}_2\cdot\vec{\gamma})\cdot\gamma^0_2 + (\theta - \frac{1}{2}[\vec{x}_1\cdot\vec{\gamma},\vec{x}_2\cdot\vec{\gamma}])\cdot\gamma^0_1\gamma^0_2 : \vec{x}_1,\vec{x}_2 \in \mathsf{R}^n ,\ \theta \in \mathsf{R} \right\}$ is a subgroup of $\mathrm{Spin}(X)$.

The hypotheses of the lemma of section 3 are satisfied: $H\cap K = \{1\}$, $K \lhd \mathrm{Spin}(2,p,q)$ (it suffices to check $aKa^{-1} = K$, $a \in H$) and $\mathbf{z}(K)\cap H = \{-1,1\}$; this results in the isomorphisms of the theorem.

It remains only to study the structure of K in detail and specify the action of $\mathrm{Spin}(p,q)$ on K.

Suppose $g = 1 + (\vec{x}_1\cdot\vec{\gamma})\cdot\gamma^0_1 + (\vec{x}_2\cdot\vec{\gamma})\cdot\gamma^0_2 + (\theta - \frac{1}{2}[\vec{x}_1\cdot\vec{\gamma},\vec{x}_2\cdot\vec{\gamma}])\cdot\gamma^0_1\gamma^0_2$ and $g' = 1 + (\vec{x}'_1\cdot\vec{\gamma})\cdot\gamma^0_1 + (\vec{x}'_2\cdot\vec{\gamma})\cdot\gamma^0_2 + (\theta' - \frac{1}{2}[\vec{x}'_1\cdot\vec{\gamma},\vec{x}'_2\cdot\vec{\gamma}])\cdot\gamma^0_1\gamma^0_2$ belong to K. Then:

$$gg' = 1 + ((\vec{x}_1+\vec{x}'_1)\cdot\vec{\gamma})\cdot\gamma^0_1 + ((\vec{x}_2+\vec{x}'_2)\cdot\vec{\gamma})\cdot\gamma^0_2$$
$$+ (\theta+\theta' - \frac{1}{2}[\vec{x}_1\cdot\vec{\gamma},\vec{x}_2\cdot\vec{\gamma}] - \frac{1}{2}[\vec{x}'_1\cdot\vec{\gamma},\vec{x}'_2\cdot\vec{\gamma}] - ((\vec{x}_1\cdot\vec{\gamma})(\vec{x}'_2\cdot\vec{\gamma}) + (\vec{x}_2\cdot\vec{\gamma})(\vec{x}'_1\cdot\vec{\gamma})))\cdot\gamma^0_1\gamma^0_2 \ .$$

This looks rather a mess but simplifies to:

$$gg' = 1 + ((\vec{x}_1+\vec{x}'_1)\cdot\vec{\gamma})\cdot\vec{\gamma}^0_1 + ((\vec{x}_2+\vec{x}'_2)\cdot\vec{\gamma})\cdot\vec{\gamma}^0_2$$
$$+ (\theta + \theta' + B(\vec{x}_1,\vec{x}'_2) - B(\vec{x}_2,\vec{x}'_1))\cdot\gamma^0_1\gamma^0_2 \ .$$

To simplify still further, let $K(p,q)$ denote the group $\mathsf{R}\times\mathsf{R}^n\times\mathsf{R}^n$ with product:

$$(\theta,\vec{x}_1,\vec{x}_2)(\theta',\vec{x}'_1,\vec{x}'_2) = (\theta+\theta'+B(\vec{x}_1,\vec{x}'_2)-B(\vec{x}_2,\vec{x}'_1),\vec{x}_1+\vec{x}'_1,\vec{x}_2+\vec{x}'_2) \qquad (5.1)$$

or equivalently:

$$(\theta,\vec{x}_1,\vec{x}_2)(\theta',\vec{x}'_1,\vec{x}'_2) = (\theta+\theta'+\tilde{B}((\vec{x}_1,\vec{x}_2),(\vec{x}'_1,\vec{x}'_2)),\vec{x}_1+\vec{x}'_1,\vec{x}_2+\vec{x}'_2) \qquad (5.2)$$

where $\tilde{B}$ denotes the bilinear form on $R^n\times R^n$ whose matrix representation is $\begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$. As $\tilde{B}$ is non-zero and skew-symmetric, $K(p,q)$ is a non-trivial, central extension of $R^n\times R^n$ by R (c.f. Bargmann (1954)), and obviously $K \cong K(p,q) \cong$ a Heisenberg group (Perroud (1977)) of dimension $2n+1$.

Finally, the action of $Spin(p,q)$ on $K(p,q)$ is the obvious one, got from the semi-direct product structure of $Spin(2,p,q)$. Rewriting $Spin(2,p,q) = \{((\theta,\vec{x}_1,\vec{x}_2),a) : (\theta,\vec{x}_1,\vec{x}_2) \in K(p,q)\ ,\ a \in Spin(p,q)\}$ we have:

$$((\theta,\vec{x}_1,\vec{x}_2),a)((\theta',\vec{x}'_1,\vec{x}'_2),a')$$
$$= ((\theta+\theta'+\tilde{B}((\vec{x}_1,\vec{x}_2),(\vec{x}'_1,\vec{x}'_2)\cdot\rho(a^{-1})),(\vec{x}_1,\vec{x}_2)+(\vec{x}'_1,\vec{x}'_2)\cdot\rho(a^{-1})),aa') \qquad (5.3)$$

so the left action of $Spin(p,q)$ on $K(p,q)$ is:

$$a\cdot(\theta,\vec{x}_1,\vec{x}_2) = (\theta,\vec{x}_1\cdot\rho(a^{-1}),\vec{x}_2\cdot\rho(a^{-1})) \qquad (5.4)$$

This completes the proof. □

As mentioned at the end of section 3, the mapping $\rho : Spin(2,p,q) \to SO(2,p,q)$ is not surjective and in fact has kernel of dimension 1.

Corollary I.5.2.: The kernel of $\rho : \mathrm{Spin}(2,p,q) \to SO(2,p,q)$ is $R\times\{-1,1\}$ (where $R \cong \{1+\theta\cdot\underset{1}{\gamma}^0\underset{2}{\gamma}^0 : \theta \in R\}$) and hence $R\times\{-1,1\} \to \mathrm{Spin}(2,p,q) \to \rho(\mathrm{Spin}(2,p,q)) = R^{2n} \circledS SO(p,q)$ is short exact, displaying $\mathrm{Spin}(2,p,q)$ as an extension of $R^{2n} \circledS SO(p,q)$ by $R\times\{-1,1\}$.

Proof: We have $g \in \ker\rho \Leftrightarrow g^{-1}xg = x$, $\forall x \in X \Leftrightarrow g \in Z(R_{2,p,q}) \cap \overset{+}{R}_{2,p,q} = \overset{+}{R}_{2,0,0} = R + R\cdot\underset{1}{\gamma}^0\underset{2}{\gamma}^0$ by Prop. I.1.7, and so $g = (1+\theta\cdot\underset{1}{\gamma}^0\underset{2}{\gamma}^0)a$, $a = \pm 1$. □

The special case $p = 0$, $q = 3$ is not surprisingly, of some interest. The group $R^6 \circledS SO(3)$ is known as the *isochronous Galilei group* (Lévy-Leblond (1971)) and its trivial central extension $R\times(R^6 \circledS SO(3))$ is the *static group* (Bacry, Lévy-Leblond (1968)).

We've shown that $\mathrm{Spin}(2,0,3)$ is the spin analogue of a non-trivial central extension of the isochronous Galilei group by R .

The Lie algebra $\mathit{spin}(2,0,3)$ has a basis:

$$J^A = \frac{1}{4}\epsilon^A{}_{BC}\gamma^B\gamma^C \quad , \quad K^A = \frac{1}{2}\underset{1}{\gamma}^0\gamma^A$$
$$P^A = \frac{1}{2}\underset{2}{\gamma}^0\gamma^A \quad , \quad \Theta = \frac{1}{2}\underset{1}{\gamma}^0\underset{2}{\gamma}^0 \tag{5.5}$$

and products (summation over repeated indices):

$$[J^A,J^B] = \epsilon^{AB}{}_C J^C \ , \ [J^A,K^B] = \epsilon^{AB}{}_C K^C \ , \ [J^A,P^B] = \epsilon^{AB}{}_C P^C \tag{5.6a}$$

$$[K^A,K^B] = 0 \ , \ [K^A,P^B] = \delta^{AB}\Theta \ , \ [P^A,P^B] = 0 \tag{5.6b}$$

$$[J^A,\Theta] = 0 \ , \ [K^A,\Theta] = 0 \ , \ [P^A,\Theta] = 0 \tag{5.6c}$$

Notice also that the *Carroll group* (Bacry, Lévy-Leblond (1968)) has its Lie algebra isomorphic to $\mathfrak{spin}(2,0,3)$ (Bacry, Lévy-Leblond (1968), p. 1612 Footnote).

The largest group of space-time transformations leaving invariant the free Schrödinger equation, called the *Schrödinger group* by Niederer (1972), turns out to be precisely $SO^+(2,0,3) \cong R^6 \, ⓢ \, (SL(2) \times SO(3))$ (see Notes).

Assuming the goal to be a definition of $Spin(r,p,q)$ as (at the very least) a covering space of $SO(r,p,q)$, we have failed in the case $r = 2$. For example, it would have been pleasant if $Spin(2,0,3)$ were the spin analogue of the Schrödinger group (or better, of its non-trivial, one dimensional central extension). What has happened, is that from $SO^+(2,0,3) \cong R^6 \, ⓢ \, (SL(2) \times SO(3))$, the spin group construction has ignored the $SL(2)$ part and in its place, put R (which is the group involved in the extension of $R^6 \, ⓢ \, SO(3))$, then taken the double cover. It is not at all clear how to remedy this situation; for further discussion see the Notes.

I.6. EXTENDED GALILEI GROUP: SPIN ANALOGUE

The *extended Galilei group* $G_{11}(m)$ (with real parameter m) is a one dimensional, central extension of the Galilei group $G_{10} \cong R^4 Ⓢ G_6 \cong R^4 Ⓢ (R^3 Ⓢ SO(3))$ by the additive group R. As hinted at in the last section, the multiplication law of G_{10} is as follows:

$$(b,\vec{a},\vec{v},R)(b',\vec{a}',\vec{v}',R') = (b+b',\vec{v}b'+\vec{a}+R\vec{a}',\vec{v}+R\vec{v}',RR') \tag{6.1}$$

where $(b,\vec{a}) \in R^4 = R \times R^3$, $(\vec{v},R) \in G_6$.

Multiplication in G_{11} is defined by:

$$(\theta,b,\vec{a},\vec{v},R)(\theta',b',\vec{a}',\vec{v}',R')$$

$$= (\theta+\theta'+\xi_m,b+b',\vec{v}b'+\vec{a}+R\vec{a}',\vec{v}+R\vec{v}',RR') \tag{6.2}$$

where $\xi_m = m(\frac{1}{2}|\vec{v}|^2 b' + \vec{v}\cdot R\vec{a}')$.

The importance of $G_{11}(m)$ in non-relativistic quantum mechanics was pointed out by Inönü and Wigner (1952), Wightman (1962), and it has been analyzed by Bargmann (1954) (see also Lévy-Leblond (1971), for a detailed discussion).

Bargmann's analysis shows that when $m \neq 0$, $G_{11}(m)$ is not a trivial extension of G_{10}. This means that $G_{11}(m) \not\cong R \times G_{10}$, when $m \neq 0$; of course $G_{11}(0) = R \times G_{10}$, as is plain from (6.2). In fact, every non-trivial central extension of G_{10} by R is equivalent to some $G_{11}(m)$, $m \neq 0$ (see the Notes for a summary of some of the relevant aspects of extensions).

The Lie algebra of $G_{11}(m)$ has a basis:

$$J^A_G \;,\; K^A_G \;,\; P^A_G \;,\; H_G \;,\; \Theta_G \qquad 1 \leq A,B \leq 3$$

with products:

$$[J^A_G, J^B_G] = \epsilon^{AB}{}_C \, J^C_G \;,\; [J^A_G, K^B_G] = \epsilon^{AB}{}_C \, K^C_G \;,\; [J^A_G, P^B_G] = \epsilon^{AB}{}_C \, P^C_G \tag{6.3a}$$

$$[J^A_G, H_G] = 0 \;,\; [K^A_G, K^B_G] = 0 \;,\; [K^A_G, P^B_G] = m\delta^{AB} \, \Theta_G \tag{6.3b}$$

$$[K^A_G, H_G] = P^A_G \;,\; [P^A_G, P^B_G] = 0 \;,\; [P^A_G, H_G] = 0 \tag{6.3c}$$

$$[J^A_G, \Theta_G] = [K^A_G, \Theta_G] = [P^A_G, \Theta_G] = [H_G, \Theta_G] = 0 \tag{6.3d}$$

Note that (6.3 a,b,c) with $m = 0$ define the Lie algebra of G_{10} ; (6.3d) displays the central nature of the extension $G_{11}(m)$ of G_{10} .

Since products of one-parameter subgroups fill out a neighbourhood of the identity, we may define

$$g(\theta, b, \vec{a}, \vec{v}, R) = \exp(\theta \cdot \Theta_G) \; \exp(b \cdot H_G) \; \exp(\vec{a} \cdot \vec{P}_G) \; \exp(\vec{v} \cdot \vec{K}_G) \; R \tag{6.4}$$

for $\theta, b, \vec{a}, \vec{v}$ sufficiently close to zero (where exp denotes exponential map, $\vec{a} \cdot \vec{P}_G = a^1 P^1_G + a^2 P^2_G + a^3 P^3_G$ etc.) and thereby recover the multiplication law (6.2) locally at the identity:

$$g(\theta, b, \vec{a}, \vec{v}, R) \; g(\theta', b', \vec{a}', \vec{v}', R')$$

$$= g(\theta + \theta' + \xi_m, b + b', \vec{v}b' + \vec{a} + R\vec{a}', \vec{v} + R\vec{v}', RR') \quad .$$

The actual calculation is very tedious and makes repeated use of the

Campbell-Baker-Hausdorff formula (Varadarajan (1974), pp. 114-121) and the relations (6.3).

In this section we shall show, by Lie algebra methods as opposed to direct calculations as in earlier sections, how the spin analogue $\mathrm{Spin}(G_{11}(m))$ of $G_{11}(m)$ fits into a Clifford algebra scheme. It will be seen how $\mathrm{Spin}(G_{11}(m))$ and $\mathrm{Spin}(L_{11})$, the spin group of a (necessarily) trivial central extension by the group $\mathbb{R}$ of the Poincaré group L_{10}, are related as subgroups of $\mathrm{Spin}(1,1,4)$.

A faithful, hence reducible, six dimensional representation (that induced by the spin group projection) of $\mathrm{Spin}(G_{11}(m))$ is given and its relation to the groups $SO(1,1,4)$ and $SO(2,5)$ discussed.

To indicate briefly why one <u>might</u> expect to find the Galilei spin group $\mathbb{R}^4 \circledS \mathrm{Spin}(1,0,3)$ inside $\mathrm{Spin}^+(1,1,4)$, recall (Cor. I.3.3) that $\mathrm{Spin}^+(1,1,4) \cong \mathbb{R}^5 \circledS \mathrm{Spin}^+(1,4)$. Therefore naively, one would believe $\mathbb{R}^4 \circledS \mathrm{Spin}(1,0,3) \subset \mathrm{Spin}^+(1,1,4)$, if only because $\mathbb{R}^4 \subset \mathbb{R}^5$ and $\mathrm{Spin}(1,0,3) \subset \mathrm{Spin}^+(1,4)$. There is more to it however. First of all, the embedding $\mathrm{Spin}(1,0,3) \subset \mathrm{Spin}^+(1,4)$ is rather special; secondly, one has to worry whether or not the actions, appearing implicitly in the semi-direct products, respect this embedding. We leave this point and return to it later. Just why the spin analogue of the extended Galilei group $G_{11}(m)$ should sit inside $\mathrm{Spin}^+(1,1,4)$ is somewhat obscure; this inclusion is described now.

Let γ^{-1} ("-1" is an index, not an exponent), $\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4$ be an orthonormal basis of $\mathbb{R}^{1,1,4}$ which generates $\mathbb{R}_{1,1,4}$: $(\gamma^{-1})^2 = 0$, $(\gamma^0)^2 = 1$, $(\gamma^1)^2 = \cdots = (\gamma^4)^2 = -1$ etc. Notice as a consequence, that

$\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4$ form an orthonormal basis of $R^{1,4} \subset R^{1,1,4}$, and they generate $R_{1,4} \subset R_{1,1,4}$. Moreover, $\gamma^0+\gamma^4, \gamma^1, \gamma^2, \gamma^3$ form an orthonormal basis of $R^{1,0,3} \subset R^{1,4}$ and they generate $R_{1,0,3} \subset R_{1,4}$. These Clifford inclusions induce Lie inclusions, and recalling (4.15), we have for the generators of Spin(1,0,3) :

$$\underset{G}{J}^A = \frac{1}{4} \epsilon^A{}_{BC} \gamma^B \gamma^C \quad , \quad \underset{G}{K}^A = \frac{1}{2} (\gamma^0+\gamma^4)\gamma^A \tag{6.5a}$$

with Lie products (4.16).

The remaining generators appearing in (6.3) also can be realized within $spin(1,1,4)$:

$$\underset{G}{P}^A = \frac{1}{2} \gamma^{-1} \gamma^A \; , \; \underset{G}{H} = \frac{1}{2} \gamma^{-1} \gamma^4 \; , \; \underset{G}{\Theta} = \frac{1}{2m} (\gamma^0+\gamma^4)\gamma^{-1} \tag{6.5b}$$

moreover, (6.4 a,b) satisfy the Lie relations (6.3) of $G_{11}(m)$ for $m \neq 0$. The Lie algebra defined by (6.5 a,b) is what is meant by $spin(G_{11}(m))$.

One might justifiably wonder what happens when $m = 0$, expecting to recover $spin(G_{10})$. Remarkably, $spin(G_{10})$ <u>does not</u> sit inside $spin(1,1,4)$ even though $spin(G_{11}(m))$, $m \neq 0$, does. This is discussed further in the Notes.

We characterize $spin(G_{11}(m))$, $m \neq 0$, in

<u>Proposition I.6.1.</u>: $spin(G_{11}(m)) = \{s \in spin(1,1,4) : [\underset{G}{\Theta}, s] = 0\}$.

<u>Proof</u>: Clearly, $spin(1,1,4)$ is generated by $\{\underset{G}{J}^A, \underset{G}{K}^A, \underset{G}{P}^A, \underset{G}{H}, \underset{G}{\Theta}, \gamma^0\gamma^A, \gamma^0\gamma^4\}$ for $1 \leq A \leq 3$, and equally clearly $spin(G_{11}(m)) \subset \{s \in spin(1,1,4) :$

$[\underset{G}{\Theta},s] = 0\}$. So suppose $s \in spin(1,1,4) \cap \mathrm{span}_R\{\gamma^0\gamma^A, \gamma^0\gamma^4\}$, that is let $s = a\cdot\gamma^0\gamma^4 + a_A\cdot\gamma^0\gamma^A$, and demand $[\underset{G}{\Theta},s] = 0$. One easily finds $s = 0$, proving the proposition. □

As $\mathrm{Spin}(G_{11}(m))$ is not defined a priori, the "correct" definition is motivated by Prop. I.6.1:

$\mathrm{Spin}(G_{11}(m))$ = the unique connected Lie subgroup of $\mathrm{Spin}^+(1,1,4)$ with Lie algebra $spin(G_{11}(m))$.

Another way to look at it is this. The adjoint group $\mathrm{Ad}(\mathrm{Spin}^+(1,1,4))$ of $\mathrm{Spin}^+(1,1,4)$ is isomorphic to $\mathrm{Spin}^+(1,1,4)$ because $\mathrm{Spin}^+(1,1,4)$ is connected with trivial centre (Helgason (1969), Cor. 5.2, p. 119), and consequently, we have a local isomorphism (at the identity):

$$\begin{aligned}\mathrm{Spin}(G_{11}(m)) &\cong \mathrm{Ad}(\mathrm{Spin}^+(1,1,4))_{\underset{G}{\Theta}}\\ &= \{\mathrm{Ad}(g) : g \in \mathrm{Spin}^+(1,1,4), \mathrm{Ad}(g)\cdot\underset{G}{\Theta} = \underset{G}{\Theta}\}\end{aligned}$$

We now indicate how the spin analogue of the Poincaré group L_{10} appears within $\mathrm{Spin}^+(1,1,4)$.

As in the extended Galilei situation $\gamma^{-1},\gamma^0,\gamma^1,\gamma^2,\gamma^3,\gamma^4$ will generate $R_{1,1,4}$ with orthonormality relations as before. By Cor. I.3.3, $\mathrm{Spin}(L_{10}) = \mathrm{Spin}^+(1,1,3)$ and $spin(1,1,3)$ has a basis given by:

$$\underset{L}{J}^A = \frac{1}{4}\,\epsilon^A{}_{BC}\,\gamma^B\gamma^C \quad , \quad \underset{L}{K}^A = \frac{1}{2}\,\gamma^0\gamma^A \tag{6.6a}$$

$$\underset{L}{P}^A = \frac{1}{2}\,\gamma^{-1}\gamma^A \quad , \quad \underset{L}{H} = \frac{1}{2}\,\gamma^0\gamma^{-1} \tag{6.6b}$$

and Lie products:

$$[J^A_L, J^B_L] = \epsilon^{AB}{}_C \, J^C_L \;,\quad [J^A_L, K^B_L] = \epsilon^{AB}{}_C \, K^C_L \;,\quad [J^A_L, P^B_L] = \epsilon^{AB}{}_C \, P^C_L \qquad (6.7a)$$

$$[J^A_L, H_L] = 0 \;,\quad [K^A_L, K^B_L] = -\epsilon^{AB}{}_C \, J^C_L \;,\quad [K^A_L, P^B_L] = \delta^{AB} \, H_L \qquad (6.7b)$$

$$[K^A_L, H_L] = P^A \;,\quad [P^A_L, P^B_L] = 0 \;,\quad [P^A_L, H_L] = 0 \qquad (6.7c)$$

Remembering the characterization of $\mathit{spin}(G_{11}(m))$ as a subalgebra of $\mathit{spin}(1,1,4)$, one verifies by a simple calculation that the centralizer of $\mathit{spin}(1,1,3)$ in $\mathit{spin}(1,1,4)$ (i.e. those elements of $\mathit{spin}(1,1,4)$ commuting with $\mathit{spin}(1,1,3)$) is just $\mathbb{R}\cdot\gamma^4\gamma^{-1}$. Defining:

$$\Theta_L = \frac{1}{2}\,\gamma^4\gamma^{-1} \qquad (6.6c)$$

we have:

$$[J^A_L, \Theta_L] = [K^A_L, \Theta_L] = [P^A_L, \Theta_L] = [H_L, \Theta_L] = 0 \qquad (6.7d)$$

and consequently $J^A_L, K^A_L, P^A_L, H_L, \Theta_L$ generate a Lie subalgebra $\mathit{spin}(L_{11})$ of $\mathit{spin}(1,1,4)$. Clearly L_{11} is to be thought of as the direct product $\mathbb{R}\times L_{10}$, that is, as the (necessarily) trivial one dimensional, central extension of L_{10} by $\mathbb{R}$. With this in mind, $\mathrm{Spin}(L_{11}) = \mathbb{R}\times \mathrm{Spin}(L_{10})$ is characterized within $\mathrm{Spin}^+(1,1,4)$ much as $\mathrm{Spin}(G_{11}(m))$ is:

$\mathrm{Spin}(L_{11})$ = unique connected Lie subgroup of $\mathrm{Spin}^+(1,1,4)$
with Lie subalgebra $\mathit{spin}(L_{11}) = \mathbb{R} \oplus \mathit{spin}(L_{10})$.

Further, to complete the analogy:

$$\mathrm{Spin}(L_{11}) \cong \mathrm{Ad}(\mathrm{Spin}^+(1,1,4))_{\underset{L}{\Theta}} = \{\mathrm{Ad}(g) : g \in \mathrm{Spin}^+(1,1,4), \mathrm{Ad}(g)\cdot\underset{L}{\Theta} = \underset{L}{\Theta}\} .$$

We return now to $\mathrm{Spin}(G_{11}(m))$ and derive the 6-dimensional representation of $G_{11}(m)$ induced by the spin group projection $\rho : \mathrm{Spin}(G_{11}(m)) \to G_{11}(m)$.

As mentioned in section 3, we have a right action of $\mathrm{Spin}(G_{11}(m))$ on coframes $(\gamma^i)_{i=-1}^{4}$ which induces a linear representation Λ of $\mathrm{Spin}(G_{11}(m))$ in $G_{11}(m)$:

$$(\gamma^i, g) \to \gamma^i\cdot\rho(g) = g^{-1}\gamma^i g = \Lambda(g)^i{}_j\,\gamma^j$$

where one verifies that $\Lambda(gg') = \Lambda(g)\,\Lambda(g')$. Now (6.4) parametrizes $\mathrm{Spin}(G_{11}(m))$ near the identity and as $\rho : \mathrm{Spin}(G_{11}(m)) \to G_{11}(m)$ has kernel $\{-1,1\}$, we obtain a parametrization of $G_{11}(m)$ near the identity:

$$\Lambda(\theta,b,\vec{a},\vec{v},R) = \Lambda(g(\theta,b,\vec{a},\vec{v},R))$$

$$= \Lambda(\exp\ \theta\cdot\underset{G}{\Theta})\cdot\Lambda(\exp\ b\cdot\underset{G}{H})\cdot\Lambda(\exp\ \vec{a}\cdot\underset{G}{\vec{P}})\cdot\Lambda(\exp\ \vec{v}\cdot\underset{G}{\vec{K}})\cdot\Lambda(R) \qquad (6.8)$$

First of all, we have:

(i) $g_1 = \exp(\omega\vec{n}\cdot\underset{G}{\vec{J}})$, the member of $\mathrm{Spin}(G_{11}(m))$ which projects to the rotation R (about axis $\vec{n}$, $|\vec{n}| = 1$ through angle ω); as $g_1^{-1}\gamma^{-1}g_1 = \gamma^{-1}$, $g_1^{-1}\gamma^0 g_1 = \gamma^0$, $g_1^{-1}\gamma^4 g_1 = \gamma^4$, $g_1^{-1}\gamma^A g_1 = R^A{}_B\,\gamma^B$ we have:

$$\Lambda(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.9a}$$

(ii) $g_2 = \exp(\vec{v}\cdot\vec{\mathrm{K}}) = 1 + \frac{1}{2}(\gamma^0+\gamma^4)\vec{v}\cdot\vec{\gamma}$, and by tedious computation:

$$\Lambda(g_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{1}{2}|\vec{v}|^2 & \vec{v}^t & \frac{1}{2}|\vec{v}|^2 \\ 0 & \vec{v} & 1_3 & \vec{v} \\ 0 & -\frac{1}{2}|\vec{v}|^2 & -\vec{v}^t & 1-\frac{1}{2}|\vec{v}|^2 \end{pmatrix} \tag{6.9b}$$

(iii) $g_3 = \exp(\vec{a}\cdot\vec{\mathrm{P}}_{\mathrm{G}}) = 1 + \frac{1}{2}\gamma^{-1}\vec{a}\cdot\vec{\gamma}$

$$\Lambda(g_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vec{a} & 0 & 1_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.9c}$$

(iv) $g_4 = \exp(b\cdot\mathrm{H}_{\mathrm{G}}) = 1 + \frac{1}{2}b\cdot\gamma^{-1}\gamma^4$

$$\Lambda(g_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1_3 & 0 \\ b & 0 & 0 & 1 \end{pmatrix} \tag{6.9d}$$

(v) $g_5 = \exp(\theta \cdot \underset{G}{\Theta}) = 1 + \frac{\theta}{2m} \cdot (\gamma^0 + \gamma^4)\gamma^{-1}$

$$\Lambda(g_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\theta}{m} & 1 & 0 & 0 \\ 0 & 0 & 1_3 & 0 \\ -\frac{\theta}{m} & 0 & 0 & 1 \end{pmatrix} \tag{6.9e}$$

where the matrices are 6×6 and in blocks; rows and columns are labelled $-1,0,1,2,3,4$; $\vec{a}$, $\vec{v}^t$ stand for 3×1 and 1×3 matrices respectively; ^{t} denotes transpose.

Multiplying, $\Lambda(g) = \Lambda(g_5)\cdot\Lambda(g_4)\cdot\Lambda(g_3)\cdot\Lambda(g_2)\cdot\Lambda(g_1)$:

$$\Lambda(\theta,b,\vec{a},\vec{v},R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\theta}{m} & 1 + \frac{1}{2}|\vec{v}|^2 & \vec{v}^t R & \frac{1}{2}|\vec{v}|^2 \\ \vec{a} & \vec{v} & R & \vec{v} \\ b - \frac{\theta}{m} & -\frac{1}{2}|\vec{v}|^2 & -\vec{v}^t R & 1 - \frac{1}{2}|\vec{v}|^2 \end{pmatrix} \tag{6.10}$$

and the multiplication law (6.2) is verified by matrix multiplication. By a change of basis, (6.10) may be replaced by the somewhat more tractable:

$$\Lambda'(\theta,b,\vec{a},\vec{v},R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\theta}{m} - \frac{b}{2} & 1 & \vec{v}^t R & \frac{1}{2}|\vec{v}|^2 \\ \vec{a} & 0 & R & \vec{v} \\ b & 0 & 0 & 1 \end{pmatrix} \tag{6.11}$$

These representations (6.10, 11) are new to the writer, although in retrospect perhaps they are not unexpected. In (6.10, 11),

the lower right 5×5 blocks provide the representation of G_6 within $S_{10} = SO^+(1,4)$ obtained by the embedding of $Spin(G_6)$ within $Spin(S_{10})$.

The realizations of $Spin(G_{11}(m))$ in $Spin^+(1,1,4)$ and $G_{11}(m)$ in $SO^+(1,1,4)$ are more significant in the light of the work of Doebner and Hennig (1974, 1976), who classify all real simple Lie algebras containing $spin(G_{11}(m))$, $m \neq 0$, and certain non-semisimple algebras containing $spin(G_{10})$ and $spin(G_{11}(m))$, $m \neq 0$. The smallest simple orthogonal Lie algebra containing $spin(G_{11}(m))$ is, according to Doebner and Hennig, $so(2,5)$. This fits in nicely with the present work, for the Clifford algebra / spin group (corresponding to a non-degenerate orthogonal space) which minimally contains the $G_{11}(m)$ Lie algebra / spin group is $R_{2,5}$ / $Spin^+(2,5)$ (indeed $R^{1,1,4} \subset R^{2,5}$ is a minimal inclusion by Prop. I.1.1 and induces $R_{1,1,4} \subset R_{2,5}$, $Spin^+(1,1,4) \subset Spin^+(2,5)$ and therefore also $G_{11}(m) \subset SO^+(1,1,4) \subset SO^+(2,5)$).

In addition to the embedding of $G_{11}(m)$ into $SO(2,5)$, Doebner and Hennig also indicate the inclusions $G_{11}(m) \subset R^5 \circledS SO(1,4)$ (as we've already discussed, $R^5 \circledS SO(1,4) \cong Spin(1,1,4) / \{-1,1\}$) , and $G_{11}(m) \subset R^7 \circledS (SL(2) \times SO(3))$. Recalling from section 5, $SO^+(2,0,3) \cong R^6 \circledS (SL(2) \times SO(3))$ and therefore $G_{11}(m) \subset R \circledS SO^+(2,0,3)$ as might have been expected according to the realization $G_{10} \subset SO^+(2,0,3)$:

$$G_{10} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ \vec{a} & \vec{v} & R \end{pmatrix} : \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in SL(2) \ , \ R \in SO(3) \ , \ (\vec{a}\ \vec{v}) \in R^6 \right\} .$$

Doebner and Hennig (1974), also ask for the construction of a dynamical algebra with $G_{11}(m)$ and L_{10} embeddings such that the embedded algebras intersect in the Euclidean Lie algebra $R^3 \circledS \mathfrak{so}(3)$. A minimal such orthogonal algebra is provided by $\mathfrak{spin}(1,1,4)$ with Euclidean subalgebra generated by $\{J^A_G = J^A_L , P^A_G = P^A_L\}$ (see (6.5, 6)).

A final incoherent remark. Because there does not seem to be any orthogonal space whose corresponding spin group is $\mathrm{Spin}(G_{11}(m))$, and given the realizations of $\mathrm{Spin}(G_{11}(m))$ within $\mathrm{Spin}^+(1,1,4)$ and $R \circledS \mathrm{Spin}(2,0,3)$, one ought to look for a new definition of spin group that deals better with the higher degenerate cases. Perhaps such would yield a compromise between $R^5 \circledS \mathrm{Spin}(2,0,3)$ and $\mathrm{Spin}^+(1,1,4)$.

I.7. NOTES AND REFERENCES

§1.

Clifford algebras have been around for some time. Dieudonné (1963), cites the 1870's work of Clifford (1876, 1878) and Lipschitz (1886) in the 1880's as two of the earlier developments. Clifford called them geometric algebras. In fact, the theory of quaternions of Hamilton (1853) and others was well developed by 1860 before Clifford made his generalization. Moreover a related theory of "biquaternions" useful in representing motions of the plane was discussed by Clifford (1873; the popular term for a combined rotation and translation was "screw motion"). While little seems to have been done in the study of $C(X)$ for arbitrary degenerate X, a class of algebras (similar to Clifford algebras and whose generators satisfy relations (1.3)) have been introduced by Eddington (1928), and studied in detail by Landsberg (1947). Modern accounts of orthogonal spaces and Clifford algebras may be found in: Artin (1957), Dieudonné (1963), Cartan (1937), Chevalley (1954), and Porteous (1969); of Clifford algebras in: Souriau (1964) and Atiyah, Bott, Shapiro (1964). From a physical standpoint one might consult Bacry (1967) and Corson (1953).

On the dimension of a Clifford algebra, see Porteous (1969), pp. 243-5 for example.

Concerning Thm. I.1.3, more can be said (here without proof) in the following generalization:

Theorem: Let $\{\gamma^0,\gamma^1,\cdots,\gamma^n\}$ be a not necessarily linearly independent

subset of $R^{1,p,q}$, $n = p+q$, satisfying the orthogonality relations (1.3) and generating a Clifford algebra. Two cases are possible:

Case A: $\gamma^0 = 0$.

(i) $\dim C = 2^n$ if n is even or if n is odd and $p-q-1 \not\equiv 0 \pmod 4$

(ii) $\dim C = 2^{n-1}$ or 2^n if n is odd and $p-q-1 \equiv 0 \pmod 4$, according as $\gamma^1\gamma^2\cdots\gamma^n$ is real or not.

Case B: $\gamma^0 \neq 0$ and we have Thm. I.1.3 (note that the relations (1.3) imply $\gamma^0,\gamma^1,\cdots,\gamma^n$ linearly independent or $\gamma^0 = 0$). □

Concerning $R_{r,0,0}$ which is clearly isomorphic to the exterior algebra on R^r , we have in addition to Prop. I.1.4:

Proposition: $\overset{+}{R}_{r,0,0}$ is an abelian algebra and consequently, is a Clifford algebra only when $r \leq 2$ (and then $\overset{+}{R}_{r,0,0} = R_{r-1,0,0}$ when $1 \leq r \leq 2$). □

Note that the only orthogonal spaces $R^{r,p,q}$ that allow abelian Clifford algebras are those with $p=q=0$.

The anti-involution of reversion can be seen to be the composition in either order of the main involution and conjugation.

We explain the quantity $\mu(I,J)$ which appears in $\gamma^I\gamma^J = \mu(I,J)\cdot\gamma^{I\Delta J}$, when $0 \notin J$. After a few moments thought, one may be convinced of the truth of the following expression:

$$\mu(I,J) = (-1)^{\rho(I,J)} \prod_{i\varepsilon I\cap J} (\gamma^i)^2$$

where $\rho(I,J) = \sum_{j\varepsilon J} \rho(I,j)$, $\rho(I,j)$ = number of members of I greater than j . The factor $(-1)^{\rho(I,J)}$ comes about in the reordering of the set $I\cup J$, and $\prod_{i\varepsilon I\cap J} (\gamma^i)^2$ because each index $i\varepsilon I\cap J$ disappears in simplifying $\gamma^I\gamma^J$. This discussion is taken from Dieudonné (1963). For example: $I = \{0,2,3,4\}$, $J = \{1,2,4\}$ => $I\Delta J = \{0,1,3\}$ and $\rho(I,J) = \rho(I,1) + \rho(I,2) + \rho(I,4) = 3+2+0 = 5$ so $\mu(I,J) = (-1)^5 (\gamma^2)^2(\gamma^4)^2 = -(\gamma^2)^2(\gamma^4)^2$, which may be verified by direct calculation.

The result of Prop. I.1.5 generalizes Prop. 13.43 of Porteous (1969).

Related to Prop. I.1.6 is the fact that the *radical* (maximal, hence two-sided, nilpotent ideal) N of $R_{r,p,q}$ is the ideal generated by those basis elements of $R^{r,p,q}$ whose squares vanish. That is:

$$N = \{ \sum_{1\leq i\leq r} a_i\cdot\gamma^i : a_i \varepsilon R_{r,p,q} ; \gamma^1,\gamma^2,\cdots,\gamma^r \text{ are those members of an orthonormal basis of } R^{r,p,q} \text{ with squares zero}\}$$

N is clearly two-sided and nilpotent of index at most $r+1$; that N is maximal follows because:

(i) by its structure, $R_{p,q}$ is known (Porteous (1969), Prop. 13.27) to be semisimple, or equivalently to have radical (0) and

(ii) $R_{r,p,q} / N \cong R_{p,q}$ as can be verified directly. For background on ring theory, consult Curtis and Reiner (1962), Ch. IV.

§2.

The discovery of spin groups in their general mathematical form (at least for non-degenerate orthogonal spaces!) is usually credited to E. Cartan (1913) on the basis of his 1913 paper. Understandably at the time, these spin groups were usually regarded as "double-valued representations" of their corresponding orthogonal groups. In fact, the very construction of a spin group in terms of a Clifford algebra embodies this "double-valuedness" (which has its cleanest manifestation in the isomorphisms: $\mathrm{Spin}(X)/\{-1,1\} = SO(X)$, $\mathrm{Pin}(X)/\{-1,1\} = O(X)$, for X non-degenerate). Intuitively, the γ^i 's form a "square root" of the bilinear form B^{ij} ; square roots are unique-up-to-sign, hence the double-valuedness.

Cartan's construction (which may be found in his book (1937)) is a trifle obscure, however it is such as to give a geometric interpretation of *spinors* (those objects on which the members of the spin group act; in modern terminology, spinor space is merely the representation space of a spin group). At about the same time Cartan's book appeared, Brauer and Weyl published an important paper (1935), stressing the Clifford algebra aspects. Noteworthy was their emphasis on the Clifford algebra, introduced by taking the "square root" (à la Dirac (1928)) of the underlying quadratic form.

While Cartan developed his version of spin group (of a non-degenerate orthogonal space) as an outgrowth of his fundamental work on semisimple Lie algebras (and it should be added, in a form general enough to encompass all the semisimple orthogonal groups $O(p,q)$), the seeds of the general theory were present some forty years prior! Indeed, there is

the work of Clifford already referred to (§1), but even closer in spirit to the modern formulation are the works of Study (1890, 1891) in the early 1890's. Study obtains a representation of the group of Euclidean motions of R^3 in a form amazingly close to that of Cor. I.3.6. It was a source of dismay for the author to have found it in Study's paper after having re-discovered it in the form of Cor. I.3.6. One can only speculate on the outcome had Cartan known of (or if he did know of it, had he further developed) the work of Clifford, Study and others (see also Buchheim (1885)).

Briefly then, there appear to have been two lines of development.

Earlier was the quaternion-motivated work of Clifford and Study which seemed to die out near the beginning of the 1900's only to be born again in the work of Dirac (1928) and Brauer and Weyl (1935) (in a generalized form), and eventually lead to the modern developments which are now standard: Artin (1957), Atiyah-Bott-Shapiro (1964), Chevalley (1954), Dieudonné (1963), and Porteous (1969) to name only a few. Brauer and Weyl evidently did not know of Study's papers, for if they had, then they would likely have obtained some results similar to those of this thesis; they probably had no good reason to consider singular quadratic forms. Coincidentally, also in 1935, there appeared a book on kinematics by Weiss (1935) wherein the results of Study are made somewhat more accessible; Weiss however seems to have been unaware of the relation between biquaternions and degenerate bilinear forms. Study's work lives on, and in unchanged form, in the 1950's textbook (also on kinematics) by Blaschke and Müller (1956).

The other main line of research was initiated by Cartan as a byproduct of his work on Lie algebras. His approach though, was geometrical (as noted previously) and sufficiently remote from the other mathematics of the day as never to have been in vogue. Today it is virtually unknown, but like much of Cartan's work, probably deserves a close re-examination. Incidentally, Penrose's geometrical interpretation (1968) of spinors is in somewhat the same spirit as Cartan's and a detailed comparison might well prove interesting.

A comment on right actions. If Φ denotes the mapping $\Gamma(X)\times X \to X$ by $(g,x) \to g^{-1}xg^{\wedge}$, then for $g\cdot g' \in \Gamma(X)$ and $x \in X$ we have:

$$\Phi(g',\Phi(g,x)) = g'^{-1}\,\Phi(g,x)g'^{\wedge} = g'^{-1}g^{-1}xg^{\wedge}g'^{\wedge}$$

$$= (gg')^{-1}\,x(gg')^{\wedge} = \Phi(gg',x)$$

This defines Φ as a right action of $\Gamma(X)$ on X. If (x_i) are the components of $x \in X$ with respect to an orthonormal basis (γ^i), and if $\Lambda = \rho_x(g)$ is the linear mapping of X induced by g, then $\gamma^i \to \Lambda^i{}_j\,\gamma^j$ and $x\cdot\rho_x(g) = (x_j\Lambda^j{}_i)$ in components relative to (γ^i).

Finally, we add a few remarks on the semi-orientations of $R^{p,q}$. If $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(p,q)$, $\Lambda \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix} \Lambda^t = \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix}$ =>

$aa^t = 1_p + bb^t$, $dd^t = 1_q + cc^t$, $c = db^t a^{-1\,t}$ and hence

$$\Lambda = \begin{pmatrix} 1_p & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1_p & b \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} 1_p & 0 \\ b^t & 1_q \end{pmatrix} \begin{pmatrix} a^{-1t} & 0 \\ 0 & 1_q \end{pmatrix} \qquad (7.1)$$

with the result that $\pm 1 = \det \Lambda = \det(d) \cdot \det(a^{-1t})$, or $\det(d) = \pm \det(a)$. Now $SO(p,q)$ consists of those $\Lambda \in O(p,q)$ such that $\det(d) \cdot \det(a^{-1t}) = 1$ i.e. $\det(d) = \det(a)$; and if $\Lambda \in SO(p,q)$ preserves the orientations on R^p and R^q, we have in addition: $\det(a) > 0$, $\det(d) > 0$. Thus $SO^+(p,q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(p,q) : \det(a) > 0 , \det(d) > 0 \right\}$ and the decomposition (7.1) shows the connectedness of $SO^+(p,q)$ to the identity (see Porteous (1969), p. 427, for $Spin^+(p,q)$).

§3.

To see why (3.2) is true, it is more convenient to work in a basis in which $g = (g^{ij})$ has the form $g = \begin{pmatrix} & & 1 \\ & 1_3 & \\ 1 & & \end{pmatrix}$. This may be achieved by a change of basis, and with respect to such a new basis $\gamma^0 + \gamma^4 = (0\ 0\ 0\ 0\ \sqrt{2})$. A tedious but straightforward calculation using matrices then shows $SO^+(1,0,3) \cong SO^+(1,4)_{\gamma^0+\gamma^4}$. Equation (3.1) follows directly without any need of a basis change.

The fact that every member of $Pin(p,q)$ is a product of members of $\mathsf{R}^{p,q} \subset \mathsf{R}_{p,q}$ is intimately related to the corresponding result about $O(p,q)$. This theorem for $O(p,q)$, due to E. Cartan and J. Dieudonné (see Dieudonné (1963), §6 - §11; Artin (1957), Thm. 3.2.0; Chevalley (1954), I.5.1), is central in the theory of orthogonal groups and says roughly that every orthogonal transformation is a product of reflections in hyperplanes with non-isotropic orthogonal complements. A special orthogonal transformation (preserving orientation) is a product

of an even number of such reflections. The relation to $Pin(p,q)$ comes about because if (γ^i) is an orthonormal basis of $R^{p,q}$ generating $R_{p,q}$, then $\gamma^i \in Pin(p,q)$ and $\rho(\gamma^i) \in O(p,q)$ is the reflection sending $(x_1,\cdots,x_i,\cdots,x_n)$ to $(x_1,\cdots,-x_i,\cdots,x_n)$, $n = p+q$.

§4.

Chevalley (1954, 2.9) discusses the Lie algebra structure of $\Gamma(X)$ for X non-degenerate, and even some of the global structure of $Pin(X)$.

As a rule, $Spin^+(r,p,q)$ is not simply connected. We deal only with $r \leq 1$; in fact, by Cor. I.3.2, $Spin^+(1,p,q)$ is homeomorphic to $R^n \times Spin^+(p,q)$ with the result that the homotopy properties of $Spin^+(1,p,q)$ are exactly those of $Spin^+(p,q)$. If $X = R^{0,0}$, $R^{1,0}$, $R^{0,1}$, $R^{1,1}$ then (Porteous (1969), p. 427) $Spin^+(p,q) = \{-1,1\}$, $\{-1,1\}$, $\{-1,1\}$, $\{-1,1\} \times R$ respectively. For all other $R^{p,q}$, $Spin^+(p,q)$ is connected and we concentrate on these cases (i.e. at least one of $p,q \geq 2$). We determine the homotopy groups $\pi_1(SO^+(p,q))$ and then use the fact that $Spin^+(p,q)$ is a double covering of $SO^+(p,q)$ to determine $\pi_1(Spin^+(p,q))$. By the Iwasawa decomposition of $SO^+(p,q)$ (Helgason (1969), Ch. VI, Thm. 5.1), $\pi_1(SO^+(p,q))$ is the same as $\pi_1(K)$, where K is a maximal compact subgroup of $SO^+(p,q)$; in this case, $K \cong SO(p) \times SO(q)$ (Wolf (1967), p. 341). Thus $\pi_1(SO^+(p,q)) = \pi_1(SO(p)) \times \pi_1(SO(q))$. Now $\pi_1(SO(n)) = \{1\}$, Z, Z_2 according as $n=1$, $n=2$, $n>2$ (Steenrod (1951), §22), and therefore (for $(p,q) \neq (0,0),(1,0),(0,1),(1,1)$):

$\pi_1(SO^+(p,q)) = Z_2$, $Z_2 \times Z_2$, Z , $Z_2 \times Z$, $Z \times Z$ according as $p=1$, $q>2$ or $p>2$, $q=1$ in the case of Z_2 ; $p,q > 2$ in the case of $Z_2 \times Z_2$; $p=1$, $q=2$ or $p=2$, $q=1$ in the case of Z ; $p=2$, $q>2$ or $p>2$, $q=2$ in the case of $Z_2 \times Z$; and $p=q=2$ in the case of $Z \times Z$. Thus $\pi_1(Spin^+(p,q))$ is simply, doubly or infinitely connected according as $p=1$, $q>2$ or $p>2$, $q=1$; $p,q > 2$; at least one of p,q equals 2 respectively.

§5.

The so-called Schrödinger group seems to have been considered first by Hagen (1972). Its structure has been discussed by Perroud (1977), in particular the isomorphism with $R^6 \circledS (SL(2) \times SO(3))$. Niederer (1974) considers the Schrödinger group $Sch(n) = R^{2n} \circledS (SL(2) \times SO(n))$ which is isomorphic to $SO^+(2,0,n)$, while Burdet, Perroud, Perrin (1978) claim the "true quantum mechanical Schrödinger group" to be essentially $SO^+(2,0,n)$ but with $SL(2)$ replaced by its double covering (this is still for spin zero).

At this point it's probably worth commenting that from quantum mechanical considerations (spin up, spin down) a spin group ought to be a double covering of a "spinless" group, but from the standpoint of unitary ray representations it would be nice if a spin group were simply connected (c.f. Bargmann (1954)). As just mentioned (§4), $Spin^+(r,p,q)$ for $r \leq 1$, is simply connected only when $p=1$, $q>2$ or $p>2$, $q=1$ (e.g. $Spin(L_6)$, $Spin(G_6)$ are simply connected. However, one physically significant case where the spin and universal covering groups differ is for $SO^+(2,4)$, the conformal group.

In wondering how to modify the definition of $\mathrm{Spin}(X)$ to achieve a more satisfactory structure when X is degenerate, one thinks immediately of the following:

(i) maintain $\mathrm{Spin}(X) \subset \Gamma(X)$ but change the condition $N(g) = \pm 1$ (this is doomed because $g \in \Gamma(X) \Rightarrow N(g) \in R$ and alteration of g by a factor is inconsequential generally).

(ii) abandon $\mathrm{Spin}(X) \subset \Gamma(X)$ choosing a possibly larger subgroup of $C(X)$ that acts "naturally" on X. (This is probably the way to go, for when X is degenerate, $C(X)$ is non-semisimple as also is $SO(X)$ and consequently automorphisms of these objects need not be inner; whereas in contrast, $\Gamma(X)$ acts on X essentially by conjugation i.e. inner automorphisms.)

No good ideas are forthcoming.

§6.

As promised in section 3, we discuss various matters related to semi-direct products. A *semi-direct product* of a group H by a group K is a group denoted $K \circledS H$, and an action of H on K (precisely, there is a homomorphism $\chi : H \to \mathrm{Aut}\, K$) such that $K \circledS H$ (also written as $K \otimes H$ to emphasize χ) is the group $(K \times H, \cdot)$:

$$(k,h)\cdot(k',h') = (k\ \chi(h)(k'),hh') \qquad (7.2)$$

The *direct product* (denoted simply $K \times H$) is obtained by taking χ to be the trivial homomorphism: $\chi(h) = \mathrm{id}_{\mathrm{Aut}\ K}$. Note that

$\tilde{K} = \{(k,1_H) : k \in K\}$, $\tilde{H} = \{(1_K,h) : h \in H\}$ are subgroups of $K \circledS H$ with $\tilde{K}$ normal. Further, $K \circledS H = \tilde{K}\cdot\tilde{H}$, $\tilde{K}\cap\tilde{H} = \{1_{K \circledS H}\}$, and the action of H on K is recovered by conjugation of $\tilde{K}$ by $\tilde{H}$ (for $(1_K,h)\cdot(k,1_H)\cdot(1_K,h)^{-1} = (\chi(h)(k),1_H)$). This characterizes a semi-direct product: if G is a group with subgroups K,H such that (i) $G = K\cdot H$ (ii) $H\cap K = \{1_G\}$ (iii) K is normal in G , then $G = K \text{ⓧ} H$ where $\chi(h)(k) = hkh^{-1}$.

Supposing then that G has subgroups K,H satisfying the conditions (i), (ii), (iii) with $\chi \in \mathrm{Hom}(H,\mathrm{Aut}\ K)$ as above, the kernel of χ is easily seen to be $z(K)\cap H = \{h \in H : kh = hk , \forall k \in K\}$. Then $z(K)\cap H \lhd G = K\cdot H$ and: $G/z(K)\cap H = K \circledS (H/z(K)\cap H)$ as well as $G = K \circledS H$. The point to all this is that the action of H on K is not *effective* ($\chi(h)(k) = k$, $\forall k \in K \Rightarrow h = 1_H$) if $z(K)\cap H \neq \{1_H\}$, but the induced action of $H/z(K)\cap H$ on K <u>is</u> effective (see Jansen, Boon (1967), I.6 and p. 75, ex. 31, for more discussion).

We now turn to the matter of extensions of groups. Given groups K,H the group E is an *extension of* H *by* K if there is an exact sequence of homomorphisms i , p :

$$K \xrightarrow{i} E \xrightarrow{p} H$$

with i one-to-one and p onto (exactness at E means $\mathrm{im}(i) = \ker(p)$ and hence by identification $E/K \cong H$). Notice that the action of E on $i(K)$ by conjugation induces an action of E on K . An extension E' of H by K is *equivalent* to E if there is an isomorphism of E to E' that respects the exact sequences defined by E,E' . The extension is *inessential* or *trivial* if there is a homomorphism $j : H \to E$

such that $p(j(h)) = h$, $\forall h \in H$; the sequence is then said to *split to the right* and E is called a *split extension.*

A basic result is the following:

Proposition: The extension $K \xrightarrow{i} E \xrightarrow{p} H$ is trivial (splits) if and only if E is a semi-direct product of H by K .

Proof: (sketch): If the sequence splits, define $\chi \in \mathrm{Hom}(H, \mathrm{Aut}\ K)$ by $i(\chi(h)(k)) = j(h)\ i(k)\ j(h^{-1})$; then $E \cong K \otimes_\chi H$. Conversely if $E = K \otimes_\chi H$, define $i(k) = (k, 1_H)$, $p(k,h) = h$, and $j(h) = (1_K, h)$ to establish that the extension splits. □

A case of particular interest is that of abelian K , for given an extension $K \xrightarrow{i} E \xrightarrow{p} H$, one readily verifies that K is an H - module (with action χ of H on K defined by: $i(\chi(h)(k)) = x\ i(k)x^{-1}$, where $x \in p^{-1}(h) \subset E$) .

A *central extension* $K \xrightarrow{i} E \xrightarrow{p} H$ (K is now necessarily abelian) is defined by the requirement that $i(K)$ be a subgroup of $Z(E)$, the centre of E ; in this case, the action of H on K is trivial and therefore, by the Proposition, a central extension is trivial if and only if it is a direct product. This explains what is meant when it's said that $G_{11}(m) \ncong \mathbb{R}\times G_{10}$ for $m \neq 0$, but $G_{11}(0) \cong \mathbb{R}\times G_{10}$.

Why do we care about central extensions? If a quantum system has a symmetry group G , then there is an associated ray representation in the Hilbert space of the system; that is (assuming a unitary ray representation), we have for $g, g' \in G$, unitary operators U_g , $U_{g'}$ and

a complex number $\omega(g,g')$ of modulus one such that:

$$U_g \, U_{g'} = \omega(g,g') \, U_{gg'} \tag{7.3}$$

Matters are simplified if we suppose a priori that $\omega(g,g') = e^{i\xi(g,g')}$, with $\xi : G\times G \to \mathrm{R}$. We want to extend the group of unitary operator rays $\{\tau\cdot U_g \; : \; |\tau| = 1\}$, to a bona fide unitary group of operators $\{\tilde{U}_{(\theta,g)}\}$ on an extended group $\tilde{G}$ of G by R, in such a way that $\tilde{U}$ is a representation of $\tilde{G}$ extending the ray representation U:

$$\tilde{U}_{(\theta,g)} \, \tilde{U}_{(\theta',g')} = \tilde{U}_{(\theta,g)(\theta',g')} \tag{7.4}$$

with $\tilde{U}_{(\theta,g)} = e^{i\theta} \, U_g$ and multiplication in $\tilde{G}$: $(\theta,g)(\theta',g') = (\theta+\theta'+\xi(g,g'),gg')$. It happens that the central extension $\tilde{G}$ so defined, is trivial essentially if $\xi(g,g') = \zeta(gg') - \zeta(g) - \zeta(g')$ for some $\zeta : G \to \mathrm{R}$ (then $\tilde{G} = \mathrm{R}\times G$, direct product, and $g \to e^{i\zeta(g)}U_g$ is a unitary representation). For details see Bargmann (1954), Moore (1970), Mackey (1978) and Lévy-Leblond (1971). For background on extensions generally, see Cartan, Eilenberg (1956), Ch. 14; Hilton, Stammbach (1971), Ch. 6, 7 and especially ex. 6.10.1 for a general treatment of multiplier functions (as in (7.3)) for non-central extensions.

As mentioned earlier, $spin(G_{11}(m)) \subset spin(1,1,4)$ for $m \neq 0$, but $spin(G_{10}) \not\subset spin(1,1,4)$. This explains why the limiting case $m = 0$ in (6.5) is, in principle, not feasible (no matter how we represent $spin(G_{11}(m))$ as a subalgebra of $spin(1,1,4)$, the limit $m \to 0$ will always be singular). The impossibility of embedding $spin(G_{10})$ in $spin(1,1,4)$ is shown as follows: the relations (6.3 a,b,c) with $m = 0$ will be shown to be untenable within $spin(1,1,4)$.

The generators J^A , $1 \le A \le 3$, determine a basis of $\mathfrak{so}(3) = \mathfrak{spin}(0,3) \subset \mathfrak{spin}(1,1,4)$, and therefore generate a subgroup of $\mathrm{Spin}^+(1,1,4)$ isomorphic to $\mathrm{Spin}(0,3)$. But then $J^A = \frac{1}{4}\epsilon^A{}_{BC}\gamma^B\gamma^C$, for some orthonormal basis $\gamma^1,\gamma^2,\gamma^3$ of $R^{0,3} \subset R^{1,1,4}$ generating $R_{0,3} \subset R_{1,1,4}$. Extend $\gamma^1,\gamma^2,\gamma^3$ to an orthonormal basis $\gamma^{-1},\gamma^0,\gamma^1$, $\gamma^2,\gamma^3,\gamma^4$ of $R^{1,1,4}$ which generates $R_{1,1,4}$. Next determine the form of K^A , given that $[J^A,K^B] = \epsilon^{AB}{}_C K^C$, $[K^A,K^B] = 0$; that is we must have (summation convention used):

$$K^A = \lambda^A{}_B J^B + \mu^A{}_B\gamma^0\gamma^B + \nu^A{}_B\gamma^4\gamma^B + \rho^A{}_B\gamma^{-1}\gamma^B$$

$$+ \sigma^A\gamma^0\gamma^4 + \tau^A\gamma^{-1}\gamma^0 + \omega^A\gamma^{-1}\gamma^4 \tag{7.6}$$

for constants $\lambda^A{}_B$, $\mu^A{}_B$, $\nu^A{}_B$, $\rho^A{}_B$, σ^A , τ^A , ω^A to be partially determined. Demanding that $[J^A,K^B] = \epsilon^{AB}{}_C K^C$, we find:

$$\lambda^A{}_B = \lambda\cdot\delta^A{}_B\ ,\quad \mu^A{}_B = \mu\cdot\delta^A{}_B\ ,\quad \nu^A{}_B = \nu\cdot\delta^A{}_B\ ,\quad \rho^A{}_B = \rho\cdot\delta^A{}_B \tag{7.7a}$$

$$\sigma^A = \tau^A = \omega^A = 0 \tag{7.7b}$$

and so $K^A = \lambda\cdot J^A + (\mu\cdot\gamma^0 + \nu\cdot\gamma^4 + \rho\cdot\gamma^{-1})\gamma^A$. The requirement $[K^A,K^B] = 0$ then implies:

$$\lambda = 0\ ,\quad \mu = \pm\,\nu \tag{7.8}$$

Let us put $\nu = \mu\,\alpha$, $\alpha = \pm 1$. Then we have:

$$K^A = (\mu\cdot(\gamma^0+\alpha\cdot\gamma^4) + \rho\cdot\gamma^{-1})\gamma^A \tag{7.9}$$

A similar argument for P^A (the P^A 's satisfy the same Lie products with themselves and the J^A 's as do the K^A 's) shows that, again $\alpha' = \underline{+}1$:

$$P^A = (\mu'\cdot(\gamma^0+\alpha'\cdot\gamma^4) + \rho'\cdot\gamma^{-1})\gamma^A \tag{7.10}$$

Finally, if J^A , K^A , P^A are to form part of a basis for $\mathit{spin}(G_{10})$, then $[K^A,P^B] = 0$ (as opposed to (6.3b) for $\mathit{spin}(G_{11}(m))$). By direct calculation, the conditions that $[K^A,P^B] = 0$, imply:

$$\mu\mu'(1-\alpha\alpha') = 0 \qquad \text{from} \qquad A \neq B \tag{7.11a}$$

$$\mu\mu'(\alpha'-\alpha) = 0 \ , \ \rho\mu' = \rho'\mu \ , \ \rho\mu'\alpha' = \rho'\mu\alpha \tag{7.11b}$$

$$\text{from} \qquad A = B \ .$$

Equations (7.11) are <u>incompatible</u> with linear independence of P^A and K^A , and consequently $\mathit{spin}(G_{10})$ is not realizable within $\mathit{spin}(1,1,4)$. Another facet of this non-realizability can be seen by setting $\theta = 0$ in (6.10, 11): the matrices $\Lambda(0,b,\vec{a},\vec{v},R)$ are not closed under multiplication (indeed, if they were, then we would have a splitting of the extension $R \to G_{11}(m) \to G_{10}$ which would imply that $G_{11}(m) = R \times G_{10}$, something we know to be false).

For completeness we now derive the most general embedding $\mathit{spin}(G_{11}(m)) \subset R_{1,1,4}$. We have:

$$J^A = \frac{1}{4}\,\epsilon^A{}_{BC}\gamma^B\gamma^C \tag{7.12a}$$

$$K^A = (\mu\cdot(\gamma^0+\alpha\cdot\gamma^4) + \rho\cdot\gamma^{-1})\gamma^A \tag{7.12b}$$

$$P^A = (\mu'\cdot(\gamma^0+\alpha'\cdot\gamma^4) + \rho'\cdot\gamma^{-1})\gamma^A \tag{7.12c}$$

where $\alpha^2 = (\alpha')^2 = 1$ and $\mu\mu'(\alpha'-\alpha) = 0$. Taking into account the relations $[J^A, H] = 0$, H is restricted to be of the form:

$$H = \sigma\cdot\gamma^{-1}\gamma^0 + \tau\cdot\gamma^{-1}\gamma^4 + \omega\cdot\gamma^0\gamma^4 \tag{7.12d}$$

Now $[K^A, H] = P^A$ and $[P^A, H] = 0$ imply that

$$\mu' = 2\mu\alpha\omega = 2\mu\alpha'\omega \ , \ \rho' = 2\mu(\alpha\tau-\sigma)$$

and

$$\mu'\omega = 0 \ , \ \mu'(\alpha'\tau-\sigma') = 0$$

respectively. Thus $\omega = 0$, $\mu' = 0$ and therefore:

$$P^A = 2\mu(\alpha\tau-\sigma)\gamma^{-1}\gamma^A \tag{7.13c}$$

$$H = \sigma\cdot\gamma^{-1}\gamma^0 + \tau\cdot\gamma^{-1}\gamma^4 \tag{7.13d}$$

$$\Theta = \frac{1}{m}[K^A, P^A] = \frac{4\mu^2}{m}(\alpha\tau-\sigma)(\gamma^0+\alpha\cdot\gamma^4)\gamma^{-1} \tag{7.13e}$$

where μ, α, ρ, σ, τ are arbitrary except that $\alpha = \pm 1$, $\mu(\alpha\tau-\sigma) \neq 0$. Thus:

$$J^A = \frac{1}{4}\epsilon^A{}_{BC}\gamma^B\gamma^C \tag{7.14a}$$

$$K^A = (\mu\cdot(\gamma^0+\alpha\gamma^4) + \rho\cdot\gamma^{-1})\gamma^A \tag{7.14b}$$

$$P^A = 2\mu(\alpha\tau-\sigma)\cdot\gamma^{-1}\gamma^A \tag{7.14c}$$

$$H = \sigma\cdot\gamma^{-1}\gamma^0 + \tau\cdot\gamma^{-1}\gamma^4 \tag{7.14d}$$

$$\Theta = \frac{4\mu^2}{m}(\alpha\tau-\sigma)\cdot(\gamma^0+\alpha\gamma^4)\gamma^{-1} \tag{7.14e}$$

Choosing $\alpha = 1$, $\mu = \frac{1}{2}$, $\rho = 0$, $\tau = \frac{1}{2}$, $\sigma = 0$ gives (6.5).

Doebner and Hennig (1974) are not the only others to have noticed that $G_{11}(m) \subset O(2,5)$. Boyer and Peñafiel (1976) noted that the Hamilton-Jacobi equation:

$$\frac{1}{2m} |\nabla S|^2 + \partial_t S = 0 \quad ; \quad S : R \times R^n \to R$$

has a symmetry group $O(2,n+2)$ which, for $n = 3$, reduces to $O(2,5)$. Furthermore, they comment without physical interpretation, that $G_{11}(m) \subset O(2,5)$.

CHAPTER II

EXPLICIT REALIZATIONS

II.0. INTRODUCTION

Whereas in Chapter I we were concerned more with the abstract notions of Clifford algebras and spin groups, we turn here to the question of matrix representations.

In order to deal effectively with Clifford algebras corresponding to real orthogonal spaces of arbitrary dimension and signature, one needs concepts general enough to be applicable when dealing with non-commutative fields (for example, the quaternions). The basic algebraic tools necessary for concrete realizations of Clifford algebras and spin groups are reviewed in section 1. Following this, computations are carried out for the groups of particular interest, and in section 3 the matrix forms of the Lie algebras of these groups are provided. Section 4 contains a detailed consideration of the four-dimensional complex representations of the Galilei Clifford algebra $R_{1,0,3}$, and indicates their relation to those of the de Sitter Clifford algebra. The chapter closes with a section of references and notes.

Whereas the proofs of the proved propositions of this chapter are the author's, only Thm. II.2.7, Cor. II.2.8, Prop. II.4.3, and Cor. II.4.6 are claimed to be new. Credit for the others is given, where possible; although no claim is made on the rest, the development is not readily available in the literature.

II.1. ALGEBRAIC PRELIMINARIES

As an explanatory remark, we mention the fact, that the Clifford algebras $\mathbb{R}_{p,q}$, corresponding to non-degenerate real orthogonal spaces, are isomorphic to matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (reals, complexes or quaternions respectively). Because $\mathbb{H}$ is non-commutative, some care has to be taken in the definition of linear spaces, mappings, etc. related to $\mathbb{H}$.

In this chapter, $\mathbb{K}$ will denote either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (as usual, $\mathbb{H}$ is the real associative algebra generated by a basis $\{1,i,j,k\}$ subject to the relations $i^2 = j^2 = k^2 = -1 = ijk$).

The symbol $\mathbb{A}$ will denote some $\mathbb{K}^s = \mathbb{K}\times\mathbb{K}\times\cdots\times\mathbb{K}$ (s times), to be regarded as a $\mathbb{K}$ - *linear algebra* (we write $\mathbb{A} = {}^s\mathbb{K}$). This means that:

$$(k_1,k_2,\cdots,k_s)+(k_1',k_2',\cdots,k_s') = (k_1+k_1',k_2+k_2',\cdots,k_s+k_s') \quad (1.1a)$$

$$(k_1,k_2,\cdots,k_s)(k_1',k_2',\cdots,k_s') = (k_1k_1',k_2k_2',\cdots,k_sk_s') \quad (1.1b)$$

defining the ring structure of $\mathbb{A}$ as that of the direct product, and for the algebra structure over $\mathbb{K}$:

$$(k_1,k_2,\cdots,k_s)\cdot k = (k_1k,k_2k,\cdots,k_sk) \quad (1.1c)$$

$$k\cdot(k_1,k_2,\cdots,k_s) = (kk_1,kk_2,\cdots,kk_s) \quad (1.1d)$$

$$a\cdot(kk') = (a\cdot k)\cdot k' \quad (1.1e)$$

$$(kk')\cdot a = k\cdot(k'\cdot a) \quad (1.1f)$$

where $k, k', k_1, \cdots, k_1', \cdots$ all belong to K and $a \in A$.

A *right* (*left*) A - *linear space* is a right (left) A - module X, such that the right (left) K - linear spaces $X\cdot(1,0,\cdots,0)$, $X\cdot(0,1,0,\cdots,0)$, $\cdots$, $X\cdot(0,\cdots,0,1)$ are isomorphic (and analogously in the left K - linear case; see Notes for details). The standard example of a right A - linear space is $A^m = \{(a^1,\cdots,a^m)^t : a^i \in A\}$ where ^{t} denotes matrix transpose (members of left linear spaces are represented by row rather than column matrices for reasons that will become clear when we consider linear mappings); then the operations are componentwise addition in A^m, and componentwise right multiplication by members of A.

An R - linear mapping t of right or left A - linear spaces X to Y is said to be *semi-linear* over A if there is an automorphism or anti-automorphism $\psi : A \to A$ such that for all $x \in X$, $a \in A$:

$$t(x\cdot a) = t(x)\cdot\psi(a) \qquad (\textit{right semi-linear}) \qquad (1.2a)$$

or

$$t(a\cdot x) = \psi(a)\cdot t(x) \qquad (\textit{left semi-linear}) \qquad (1.2b)$$

when ψ is an automorphism or

$$t(x\cdot a) = \psi(a)\cdot t(x) \qquad (\textit{right-to-left semi-linear}) \qquad (1.2c)$$

or

$$t(a\cdot x) = t(x)\cdot\psi(a) \qquad (\textit{left-to-right semi-linear}) \qquad (1.2d)$$

when ψ is an anti-automorphism. Because t determines ψ (except when $t = 0$), one says that t is A^ψ - linear.

The following examples have been taken from Porteous (1969, p. 199), as has been the discussion and notation up to now. Here, $A = H = \langle 1,i,j,k \rangle$, $X=Y=H$:

(i) $t(x) = x$; $\psi(a) = a$

(ii) $t(x) = bx$, $0 \neq b \in A$; $\psi(a) = a$

(iii) $t(x) = xc$, $0 \neq c \in A$; $\psi(a) = c^{-1}ac$

(iv) $t(x) = bxc$, $0 \neq b$, $c \in A$; $\psi(a) = c^{-1}ac$

(v) $t(x) = jxj^{-1}$; $\psi(a) = jaj^{-1}$

These are examples of invertible, right semi-linear mappings, whereas the following are right-to-left semi-linear: $A=H$, $X=Y=A^2=H^2$

(vi) $t\begin{pmatrix} x \\ y \end{pmatrix} = (\overline{x},\overline{y})$; $\psi(a) = \overline{a}$

(vii) $t\begin{pmatrix} x \\ y \end{pmatrix} = (\overline{y},\overline{x})$; $\psi(a) = \overline{a}$

If X,Y are both right (left) A - linear, $L(X,Y)$ denotes the family of A - linear (i.e. A^ψ - linear with $\psi = id_A$) mappings; $L(X,Y)$ is in general neither a right nor a left A - linear space (see Notes for discussion). However, the *dual space* $X^L = L(X,A)$ of a right (left) A - linear space has the structure of a left (right) A- linear space: indeed, for $t \in X^L$, $a \in A$, $x \in X$ define $(a \cdot t)(x) = at(x)$ (respectively $(t \cdot a)(x) = t(x)a$) when X is right (respectively left) A - linear. For an A^ψ - linear map $t : X \to Y$, one verifies (by checking the four possible cases of semi-linearity) that $\psi^{-1} \circ \gamma \circ t \in X^L$ whenever $\gamma \in Y^L$. The map $t^L : Y^L \to X^L$, called the *dual* of t , defined by $t^L(\gamma) = \psi^{-1} \circ \gamma \circ t$ is $A^{\psi^{-1}}$ - linear, as may be verified case by case.

A *correlation* on a right A - linear space X is an A - semi-linear map $\xi : X \to X^L$; and the map $X \times X \to A$ by $(x,y) \to \xi(x)\cdot y = \xi(x)(y)$ is the *product* induced by the correlation (analogous to inner product). The *form* induced by the correlation is the map $X \to A$ by $x \to \xi(x)\cdot x$. Although R - bilinear, the product is generally not A - bilinear; it is, however, *sesqui-linear* i.e. right-to-left semi-linear over A (as indeed are Hermitean inner products over C). The notation $< \, , \, >_\xi$ is sometimes used to denote the product induced by ξ ($<x,y>_\xi = \xi(x)\cdot y$); using this notation, sesqui-linearity takes the form:

$$<x,y\cdot a>_\xi = <x,y>_\xi a \quad ; \quad <x\cdot a,y>_\xi = \psi(a)<x,y>_\xi \ .$$

An A^ψ - correlation $\xi : X \to X^L$ is *symmetric* or *skew* if for all $x,y \in X$:

$$<y,x>_\xi = \begin{cases} \psi(<x,y>_\xi) & \text{symmetric case} \\ -\psi(<x,y>_\xi) & \text{skew case} \end{cases} \tag{1.3}$$

A symmetric product ξ over C^ψ , H^ψ where ψ is the conjugation anti-involution ($\psi(1) = 1$, $\psi(i) = -i$ in the case of C ; $\psi(1) = 1$, $\psi(i) = -i$, $\psi(j) = -j$, $\psi(k) = -k$ in the case of H) is said to be *Hermitean*. A correlation ξ with the property that $<x,y>_\xi = 0 \iff <y,x>_\xi = 0$ is said to be *reflexive* (the most notable members of this class being the symmetric and skew correlations). An invertible correlation is said to be *non-degenerate*.

The following results (see Porteous (1969), pp. 207-210 for proofs and further discussion) are important for the computations that appear in section 2.

<u>Proposition II.1.1.</u>: Let ξ , η be non-degenerate A^{ψ} - correlations on finite dimensional right A - linear spaces X , Y respectively and let $t : X \to Y$ be A - linear (i.e. A^{ψ} - linear with $\psi = \mathrm{id}_A$). Then there is a unique A - linear map $t^* : Y \to X$ with the property that $\langle y, t(x) \rangle_{\eta} = \langle t^*(y), x \rangle_{\xi}$ for all $x \in X$, $y \in Y$. □

The map t^* is called the *adjoint* of t with respect to ξ , η and is given (as may easily be checked) by $t^* = \xi^{-1} \circ t^L \circ \eta$. In terms of diagrams, t^* is defined so that the diagram:

$$\begin{tikzcd} X \arrow[d, "\xi"'] & Y \arrow[l, dashed, "t^*"'] \arrow[d, "\eta"] \\ X^L & Y^L \arrow[l, "t^L"] \end{tikzcd}$$

commutes. If $u : Y \to Z$ is also A - linear and ζ a non-degenerate correlation on Z , then $(u \circ t)^* = t^* \circ u^*$ where u^* and $(u \circ t)^*$ are adjoints with respect to η , ζ and ξ , ζ respectively.

The next proposition is essential for the realization of Clifford algebras as algebras of linear transformations.

<u>Proposition II.1.2.</u>: Let X be a finite dimensional right A - linear space. Then any anti-involution of the real algebra $\mathrm{End}\, X = L(X,X)$ is representable as the adjoint anti-involution induced by a non-degenerate reflexive correlation on X . □

The stipulation that $L(X,X)$, when regarded as an algebra, is to be a real algebra comes about because as mentioned before, $L(X,X)$ need not be A - linear, although it will be linear over the centre of A. (For elaboration, see the Notes.)

An A^{ψ} - *correlated space* (X,ξ) is a right A - linear space X with an A^{ψ} - correlation on it, and (X,ξ) will be said to be *non-degenerate*, *reflexive*, *symmetric* or *skew* if ξ has the corresponding property. It is *totally isotropic* if ξ is zero and *neutral* if X is the direct sum of two totally isotropic subspaces (each subspace of X having the correlation induced on it by restriction of ξ).

In the following examples, ψ will be an anti-involution (an anti-automorphism with $\psi^2 = \mathrm{id}$) of A :

(i) $X = A^2$, right A - linear, with the A^{ψ} - sesqui-linear product $A^2 \times A^2 \to A$ defined by

$$<\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix}>_{hb} = \psi(b)a' + \psi(a)b' \tag{1.4}$$

is a symmetric, neutral, non-degenerate A^{ψ} - correlated space called the *standard hyperbolic plane* A^{ψ}_{hb}. To be specific, when $A = H$ there are up to isomorphism exactly two anti-involutions on H (Porteous (1969), p. 181): conjugation (with $\psi(a) = \bar{a}$), and reversion with respect to j (with $\psi(a) = j\bar{a}j^{-1}$). If $A = {}^2K$, then an important class of anti-involutions on A is the family of all ψ such that $\psi(k_1,k_2) = (\chi(k_2),\chi^{-1}(k_1))$, with $(k_1,k_2) \in {}^2K$ and χ an anti-automorphism of K (not necessarily an anti-involution).

For this A , the product is still given by (1.4), however $a = (a_1, a_2)$, $b = (b_1, b_2)$, etc.

(ii) $X = A^2$, right A - linear, with the A^{ψ} - sesqui-linear product defined by

$$<\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix}>_{sp} = \psi(b)a' - \psi(a)b' \tag{1.5}$$

is a skew, neutral, non-degenerate A^{ψ} - correlated space called the *standard symplectic plane* A^{ψ}_{sp} .

(iii) $X = A^2$, right A - linear with the A^{ψ} - sesqui-linear product defined by

$$<\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix}>_{+} = \psi(a)a' + \psi(b)b' \tag{1.6}$$

is a symmetric, non-degenerate A^{ψ} - correlated space, denoted by $(A^{\psi})^2$. It is the A^{ψ} - *product* of (A, ψ) with itself, denoted $A^{\psi} \times A^{\psi}$.

(iv) $X = A^2$, with A as before and the product defined by

$$<\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix}>_{-} = \psi(a)a' - \psi(b)b' \tag{1.7}$$

is a symmetric, neutral, non-degenerate A^{ψ} - correlated space. When $A = K$ and ψ = conjugation, this space is denoted by $\overline{K}^{1,1}$.

Further discussion of semi-linear maps and correlated spaces will be found in the Notes section.

In order not to lose sight of the basic goal of this chapter, namely to provide matrix representations of the de Sitter, Lorentz and homogeneous Galilei Clifford algebras and spin groups, we recall some facts about the Clifford algebras $R_{p,q}$.

By $A(m)$ we mean the <u>real</u> endomorphism algebra End $A^m = L(A^m, A^m)$. The following isomorphisms may be found in Porteous (1969, p. 245, pp. 248-249) and in a different formalism in Souriau (1964, §44):

(i) $R_{1,0} \cong {}^2R$ with orthonormal basis $\{(1,-1)\}$ (1.8a)

(ii) $R_{0,q} \cong R, C, H, {}^2H, H(2)$ (1.8b)

according as $q = 0,1,2,3,4$ with orthonormal subsets: ϕ for R , $\{i\}$ for C , $\{i,k\}$ for H , $\{(i,-i),(j,-j),(k,-k)\}$ for ${}^2H$, and $\{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$ for $H(2)$.

(iii) $R_{p+1,q} \cong R_{q+1,p}$ (1.8c)

(iv) $R_{p,q+4} \cong R_{p,q} \otimes R_{0,4} \cong R_{p,q} \otimes H(2)$ (1.8d)

(v) $R_{p,q+8} \cong R_{p,q} \otimes R(16) \cong R_{p,q}(16)$ (1.8e)

Every $R_{p,q}$ is of the form $A(m)$ for A one of $R, C, H, {}^2R, {}^2H$ and A^m is called the *real spinor space* of $R^{p,q}$ and its elements the $R^{p,q}$ *spinors*. Since $A = {}^sK$, we have the decompositions: $A^m = K^m \oplus \cdots \oplus K^m$ and $A(m) = K(m) \oplus \cdots \oplus K(m)$ both with s summands, with the result that $R_{p,q}$ is semisimple (c.f. Wedderburn's Theorem, Curtis and Reiner (1962) and Notes for Ch. I. §1).

A few examples of physical importance follow (recall that $\overset{+}{\mathrm{R}}_{p,q} \cong \mathrm{R}_{p,q-1}$ when $q \geq 1$ and $\mathrm{R}_{q,p-1}$ when $p \geq 1$ by Prop. I.1.4):

(i) $\mathrm{R}^{0,3}$ (Euclidean space)

$$\mathrm{R}_{0,3} = {}^2\mathrm{H} \; ; \; \overset{+}{\mathrm{R}}_{0,3} = \mathrm{R}_{0,2} = \mathrm{H} \tag{1.9a}$$

(ii) $\mathrm{R}^{1,3}$ (Minkowski space-time)

$$\mathrm{R}_{1,3} = \mathrm{R}_{4,0} = \mathrm{H}(2) \; ; \; \overset{+}{\mathrm{R}}_{1,3} = \mathrm{R}_{1,2} = \mathrm{C}(2) \tag{1.9b}$$

(iii) $\mathrm{R}^{1,4}$ (contains de Sitter space-time as an embedded quadric)

$$\mathrm{R}_{1,4} = \mathrm{R}_{1,0} \otimes \mathrm{R}_{0,4} = {}^2\mathrm{H}(2) \; ; \; \overset{+}{\mathrm{R}}_{1,4} = \mathrm{R}_{1,3} = \mathrm{H}(2) \tag{1.9c}$$

(iv) $\mathrm{R}^{1,0,3}$ (flat Galilei space-time)

$$\mathrm{R}_{1,0,3} \subset \mathrm{R}_{1,4} = {}^2\mathrm{H}(2) \; ; \; \overset{+}{\mathrm{R}}_{1,0,3} \subset \overset{+}{\mathrm{R}}_{1,4} = \mathrm{H}(2) \tag{1.9d}$$

Of course, $\mathrm{R}_{r,p,q}$ for $r > 0$ is <u>never</u> of the form $\mathrm{A}(m)$, since $\mathrm{R}_{r,p,q}$ has a non-trivial nilpotent ideal (c.f. Notes for Ch. I. §1). The inclusion $\mathrm{R}_{1,0,3} \subset \mathrm{R}_{1,4}$ has been discussed in Ch. I. §6; the matrix statement will follow.

Since $\mathrm{R}_{p,q}$ is of the form $\mathrm{A}(m)$, its spin group $\mathrm{Spin}(p,q)$ will be a subgroup of the invertible elements of $\mathrm{A}(m)$. The conjugation anti-involution on $\mathrm{R}_{p,q}$ appears in the definition of $\mathrm{Spin}(p,q)$ and its counterpart on $\mathrm{A}(m)$ is required for the definition of $\mathrm{Spin}(p,q)$ as a subgroup of $\mathrm{A}(m)$. Now a symmetric or skew, non-degenerate correlation on the spinor space A^m induces an adjoint on $\mathrm{A}(m)$ which is, moreover, an anti-involution. Conversely, we have Prop. II.1.2 and it is essential, therefore, to know the appropriate correlation.

To this end one has (Porteous (1969), pp. 265-270):

Proposition II.1.3.: Conjugation on $R_{0,n}$ is adjoint induced by the standard positive definite correlation or spinor space A^m, where $R_{0,n} = A(m)$. □

Proposition II.1.4.: For $R_{p,q} = A(m)$ with $p > 0$ and $(p,q) \neq (1,0)$, conjugation is the adjoint on $A(m)$ induced by a neutral semi-linear correlation on A^m . □

Before proceeding with specific computations of spin groups, we illustrate the notion of adjoint by determining the adjoints associated with the four examples of correlations just given. Two of these will be used later.

(i) A^{ψ}_{hb} where $X = A^2$ and $\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \rangle_{hb} = \psi(y)x' + \psi(x)y'$:

Let $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{End } X$ (see Notes for comment on matrices).

The relation: $\langle t^* \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \rangle_{hb} = \langle \begin{pmatrix} x \\ y \end{pmatrix}, t \begin{pmatrix} x' \\ y' \end{pmatrix} \rangle_{hb}$ determines t^*. In fact, the right-hand side equals $\psi(y)(ax'+by') + \psi(x)(cx'+dy') = \psi(\psi(c)x+\psi(a)y)x' + \psi(\psi(d)x+\psi(b)y)y'$ (recall that ψ is an anti-involution) and therefore comparing t and t^* :

$$t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad t^* = \begin{pmatrix} \psi(d) & \psi(b) \\ \psi(c) & \psi(a) \end{pmatrix} \tag{1.10}$$

(ii) A^{ψ}_{sp} , $X = A^2$, $\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \rangle_{sp} = \psi(y)x' - \psi(x)y'$:

With $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\langle t^* \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \rangle_{sp} = \langle \begin{pmatrix} x \\ y \end{pmatrix}, t \begin{pmatrix} x' \\ y' \end{pmatrix} \rangle_{sp}$ whose

right-hand side is

$\psi(y)(ax'+by')-\psi(x)(cx'+dy') = \psi(-\psi(c)x+\psi(a)y)x'-\psi(\psi(d)x-\psi(b)y)y'$

so $t^*\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\psi(d)x - \psi(b)y\\-\psi(c)x + \psi(a)y\end{pmatrix}$ hence

$$t = \begin{pmatrix}a & b\\c & d\end{pmatrix} \quad , \quad t^* = \begin{pmatrix}\psi(d) & -\psi(b)\\-\psi(c) & \psi(a)\end{pmatrix} \tag{1.11}$$

(iii) $A^\psi \times A^\psi$, $X = A^2$, $<\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x'\\y'\end{pmatrix}>_+ = \psi(x)x' + \psi(y)y'$: We have

$$t = \begin{pmatrix}a & b\\c & d\end{pmatrix} \quad , \quad t^* = \begin{pmatrix}\psi(a) & \psi(c)\\\psi(b) & \psi(d)\end{pmatrix} \tag{1.12}$$

(iv) $X = A^2$, $<\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x'\\y'\end{pmatrix}>_- = \psi(x)x' - \psi(y)y'$: Then

$$t = \begin{pmatrix}a & b\\c & d\end{pmatrix} \quad , \quad t^* = \begin{pmatrix}\psi(a) & -\psi(c)\\-\psi(b) & \psi(d)\end{pmatrix} \tag{1.13}$$

II.2. DE SITTER, LORENTZ, GALILEI REALIZATIONS

As the Lorentz and homogeneous Galilei Clifford algebras (and spin groups) both are embedded in the de Sitter Clifford algebra (and spin group) we proceed first with the de Sitter case.

de Sitter

From (1.9c), (1.8 a,b), an orthonormal basis of $R^{1,4}$ generating $R_{1,4}$ can be obtained by tensoring with $(1,-1)$, the matrices $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$, $\begin{pmatrix}j & 0\\ 0 & -j\end{pmatrix}$, $\begin{pmatrix}k & 0\\ 0 & -k\end{pmatrix}$, $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, the latter four being orthonormal generators of $R_{0,4} = H(2)$. Instead of this basis of $R^{0,4}$, we use an equivalent set (similar by $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$) : $\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$, $\begin{pmatrix}0 & j\\ j & 0\end{pmatrix}$, $\begin{pmatrix}0 & k\\ k & 0\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Tensoring with $(1,-1)$, we obtain for $R_{1,4}$ an orthonormal generating set $\Gamma^0,\Gamma^1,\Gamma^2,\Gamma^3,\Gamma^4$:

$$\Gamma^0 = (1,-1) \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}(1,-1) & (0,0)\\ (0,0) & (-1,1)\end{pmatrix}$$

$$= \gamma^0 \oplus -\gamma^0 \quad \text{where} \quad \gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.1a}$$

$$\Gamma^1 = (1,-1) \otimes \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = \begin{pmatrix}(0,0) & (i,-i)\\ (i,-i) & (0,0)\end{pmatrix}$$

$$= \gamma^1 \oplus -\gamma^1 \quad \text{where} \quad \gamma^1 = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \tag{2.1b}$$

$$\Gamma^2 = \gamma^2 \oplus -\gamma^2 \quad \text{where} \quad \gamma^2 = \begin{pmatrix}0 & j\\ j & 0\end{pmatrix} \tag{2.1c}$$

$$\Gamma^3 = \gamma^3 \oplus -\gamma^3 \quad \text{where} \quad \gamma^3 = \begin{pmatrix}0 & k\\ k & 0\end{pmatrix} \tag{2.1d}$$

$$\Gamma^4 = (1,-1) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} (0,0) & (1,-1) \\ (-1,1) & (0,0) \end{pmatrix}$$

$$= \gamma^4 \oplus -\gamma^4 \quad \text{where} \quad \gamma^4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.1e}$$

Note that $\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 = 1$, and that $\gamma^0,\cdots,\gamma^4$ form an orthonormal basis of $\mathsf{R}^{1,4}$ but only generate a non-universal Clifford algebra of type $(1,4)$: $\mathsf{H}(2) = \langle\gamma^0,\gamma^1,\gamma^2,\gamma^3,\gamma^4\rangle$.

Thus, $\mathsf{R}_{1,4} = \langle\Gamma^0,\Gamma^1,\Gamma^2,\Gamma^3,\Gamma^4\rangle = {}^2\mathsf{H}(2)$ and $\overset{+}{\mathsf{R}}_{1,4} = \langle\Gamma^i\Gamma^j : 0 \le i < j \le 4\rangle = \langle\Gamma^0\Gamma^4,\Gamma^1\Gamma^4,\Gamma^2\Gamma^4,\Gamma^3\Gamma^4\rangle$. However, $\Gamma^\alpha\Gamma^4 = \gamma^\alpha\gamma^4 \oplus \gamma^\alpha\gamma^4$, $0 \le \alpha \le 3$, and consequently $\overset{+}{\mathsf{R}}_{1,4} \cong \langle\gamma^0\gamma^4,\gamma^1\gamma^4,\gamma^2\gamma^4,\gamma^3\gamma^4\rangle = \langle\gamma^0,\gamma^1,\gamma^2,\gamma^3\rangle$ recalling that $\gamma^4 = -\gamma^0\gamma^1\gamma^2\gamma^3$; thus $\overset{+}{\mathsf{R}}_{1,4} \cong \mathsf{R}_{1,3} = \mathsf{H}(2)$.

Now conjugation on $\mathsf{R}_{1,4}$ is adjoint with respect to some correlation on $({}^2\mathsf{H})^2$. Consider the correlation (1.7) where $\mathsf{A} = {}^2\mathsf{H}$ and $\psi(x_1,x_2) = (\overline{x}_2,\overline{x}_1)$, and the associated adjoint (1.13). It is easy to verify that $\Gamma^{i\,*} = -\Gamma^i$, $0 \le i \le 4$.

Turning to spin groups we recall:

$$\mathrm{Spin}(1,4) = \{t \,\varepsilon\, \overset{+}{\Gamma}(\mathsf{R}^{1,4}) : t^*t = \pm 1\}$$

and

$$\mathrm{Spin}^+(1,4) = \{t \,\varepsilon\, \overset{+}{\Gamma}(\mathsf{R}^{1,4}) : t^*t = 1\}$$

It turns out that when $p+q \le 5$, $t^*t = 1$ and $t \,\varepsilon\, \overset{+}{\mathsf{R}}_{p,q}$ already imply $t \,\varepsilon\, \Gamma(\mathsf{R}^{p,q})$ (Porteous (1969), Prop. 13.58, p. 264) with the result that:

$$\mathrm{Spin}(1,4) = \{t \in \overset{+}{\mathrm{R}}_{1,4} : t^* t = \underline{+}1\}$$

and

$$\mathrm{Spin}^+(1,4) = \{t \in \overset{+}{\mathrm{R}}_{1,4} : t^* t = 1\}$$

However, whereas $\mathrm{R}_{1,4} = {}^2\mathrm{H}(2) = \mathrm{H}(2) \oplus \mathrm{H}(2)$, $\overset{+}{\mathrm{R}}_{1,4} = (1,1) \otimes \mathrm{H}(2) \cong \mathrm{H}(2)$ and so we may think of $\mathrm{Spin}(1,4)$ as a subgroup of $\mathrm{H}(2)$. If $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{H}(2)$, then $t^* = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}$ and $t^* t = \underline{+}1 \iff |a|^2 - |c|^2 = |d|^2 - |b|^2 = \underline{+}1$, $\bar{a}b = \bar{c}d$.

This proves:

<u>Theorem II.2.1.</u>:

$$\mathrm{Spin}(1,4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{H}(2) : |a|^2 - |c|^2 = |d|^2 - |b|^2 = \underline{+}1 \ , \ \bar{a}b = \bar{c}d \right\}$$

$$\mathrm{Spin}^+(1,4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{H}(2) : |a|^2 - |c|^2 = |d|^2 - |b|^2 = 1 \ , \ \bar{a}b = \bar{c}d \right\}$$

□

This may be partially rephrased as:

<u>Corollary II.2.2.</u>: $\mathrm{Spin}^+(1,4) = \mathrm{Sp}(1,1)$ where by definition $\mathrm{Sp}(p,q) = \{g \in \mathrm{H}(p+q) : \bar{g}^t K_{p,q} g = K_{p,q}\}$ and $\bar{g}^t$ denotes the transposed conjugate of g and $K_{p,q} = \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix}$.

<u>Proof</u>: This follows at once from Thm. II.2.1 and the definition of Sp(1,1) . The notation "Sp" stands for symplectic (with quaternionic overtones, as was the original usage). □

Generally, if $R_{p,q} = A(m)$, then $Spin(p,q)$ will, by definition, be a subgroup of $A(m)$ (one first determines $\overset{+}{R}_{p,q} \subset A(m)$, then $\Gamma(\overset{+}{R}_{p,q})$ and finally those $t \in \Gamma(\overset{+}{R}_{p,q})$ such that $t^*t = \pm 1$). This is sometimes inconvenient because the size of the matrices is not minimal. What is usually done is the following:

(i) obtain an orthonormal generating set $\{\Gamma^i\}$, $1 \leq i \leq n$, for $R_{p,q} = A(m)$, $n = p+q$.

(ii) obtain an orthonormal generating set $\{\Gamma^\alpha\Gamma^n\}$, $1 \leq \alpha \leq n-1$ for $\overset{+}{R}_{p,q} = A'(m')$ with A', m' usually different from A, m respectively (c.f. Prop. I.1.4).

(iii) conjugation in $R_{p,q}$ is defined by $\Gamma^{i*} = -\Gamma^i$, and it induces a conjugation on $\overset{+}{R}_{p,q}$ since (within $R_{p,q}$) $(\Gamma^\alpha\Gamma^n)^* = -\Gamma^\alpha\Gamma^n$, $1 \leq \alpha \leq n-1$; one now determines what $(\Gamma^\alpha\Gamma^n)^* = -\Gamma^\alpha\Gamma^n$ means in $\overset{+}{R}_{p,q} = A'(m')$

(iv) determine $\Gamma(\overset{+}{R}_{p,q})$ and an adjoint (from a correlation on $(A')^{m'}$) on $A'(m')$ which sends each $\Gamma^\alpha\Gamma^n$ to its negative; then compute $Spin(p,q) \subset A'(m')$.

This is essentially what we've done here: $\Gamma^i = \gamma^i \oplus -\gamma^i$, so $\Gamma^\alpha\Gamma^4 = \gamma^\alpha\gamma^4 \oplus \gamma^\alpha\gamma^4$ is to be identified with $\gamma^\alpha\gamma^4$. Now find a correlation on $(A')^{m'}$ whose adjoint on $A'(m')$ sends $\gamma^\alpha\gamma^4$ to $-\gamma^\alpha\gamma^4$; $t^*t = \pm 1$, for $t \in A'(m')$, uses the adjoint so found on $A'(m')$ to determine $Spin(p,q)$ as a subgroup of $A'(m')$ rather than of $A(m)$.

In detail, given that $\overset{+}{R}_{1,4} = H(2) = \langle\gamma^0,\gamma^1,\gamma^2,\gamma^3,\gamma^4\rangle$, and moreover that $\{\gamma^\alpha\gamma^4\}$, $0 \leq \alpha \leq 3$, is an orthonormal generating set for

$R_{1,3} = \overset{+}{R}_{1,4}$, we must find a correlation on H^2 whose adjoint induces the mapping $\gamma^\alpha\gamma^4 \to -\gamma^\alpha\gamma^4$. The correlated space $\overline{H}^{1,1}$ does the trick (c.f. (1.7), (1.13)); with its adjoint, $\gamma^{i*} = \gamma^i$, $0 \leq i \leq 4$, and hence $(\gamma^\alpha\gamma^4)^* = -\gamma^\alpha\gamma^4$, $0 \leq \alpha \leq 3$, as required. Thus we arrive at Thm. II.2.1 and Cor. II.2.2 and circumvent the (albeit trivially more complicated) $H(4)$ representations of $Spin(1,4)$.

The correlated space $\overline{H}^{1,1}$ is not the only one that works. Consider instead, H^ψ_{sp} , with ψ being the anti-involution (of H) of reversion with respect to j (c.f. (1.5) with $A = H$). The corresponding adjoint on $H(2)$ is given by (1.11):

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \psi(d) & -\psi(b) \\ -\psi(c) & \psi(a) \end{pmatrix}$. With respect to this adjoint we have:

$\gamma^{0*} = -\gamma^0$, $\gamma^{1*} = -\gamma^1$, $\gamma^{2*} = \gamma^2$, $\gamma^{3*} = -\gamma^3$, $\gamma^{4*} = -\gamma^4$. However $\overset{+}{R}_{1,4} = R_{1,3} = \langle\gamma^0,\gamma^1,\gamma^3,\gamma^4\rangle$ and the adjoint, so constructed, on $H(2)$ is precisely conjugation on $\overset{+}{R}_{1,4}$. Consequently we have:

<u>Theorem II.2.3.</u>: If $\psi(a) = j\bar{a}j^{-1}$, $a \in H$ then:

$$Spin(1,4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(2) : \psi(a)d-\psi(c)b = \pm 1 , \; \psi(d)b = \psi(b)d , \; \psi(c)a = \psi(a)c \right\}$$

$$Spin^+(1,4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(2) : \psi(a)d-\psi(c)b = 1 , \; \psi(d)b = \psi(b)d , \; \psi(c)a = \psi(a)c \right\}$$

<u>Proof</u>: This involves examination of the condition $\overset{*}{t}t = \pm 1$. □

For further comments on Thm. II.2.1 and Thm. II.2.3 see the Notes.

Lorentz

While the isomorphism $Spin^{+}(1,3) \cong SL(2;\mathbb{C})$ is well known, we shall work within the general framework outlined. Contact is made with the $SL(2;\mathbb{C})$ representation.

From (1.9b) or from the de Sitter case just discussed, we have $R_{1,3} = H(2)$. For an orthonormal generating set of $R_{1,3}$, we use:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \qquad (2.2)$$

(these being similar through $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to γ^α, $0 \le \alpha \le 3$, of the de Sitter case). The adjoint defined by the correlated space $\overline{H}^{1,1}$ (see (1.7)), induces the mapping $\gamma^\alpha \to \gamma^{\alpha *} = -\gamma^\alpha$, $0 \le \alpha \le 3$, and is, therefore, conjugation on $R_{1,3} = H(2)$. We know, from (1.9b), that $\overset{+}{R}_{1,3} \cong C(2)$ but this is not so obvious at this stage. In fact, $\overset{+}{R}_{1,3}$ is spanned by $\{1_2, \gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3, \gamma^1\gamma^2, \gamma^2\gamma^3, \gamma^3\gamma^1, \gamma^0\gamma^1\gamma^2\gamma^3\}$. An easy computation leads to the representation: $\overset{+}{R}_{1,3} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in H(2)$.

This is enough to determine the spin groups.

Theorem II.2.4.:

$$Spin(1,3) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in H(2) : |a|^2 - |b|^2 = \pm 1 \ , \ \overline{a}b + \overline{b}a = 0 \right\}$$

$$Spin^{+}(1,3) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in H(2) : |a|^2 - |b|^2 = 1 \ , \ \overline{a}b + \overline{b}a = 0 \right\}$$

Proof: Recall the remarks leading up to the statement of Thm. II.2.1; here we have $c = -b$, $d = a$. □

In fact, if we let $\gamma^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then the γ^α 's , $0 \le \alpha \le 3$ of (2.2) along with this γ^4 , generate the non-universal Clifford algebra $\mathrm{H}(2)$, for $\mathrm{R}^{1,4}$. Moreover, $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\ \mathrm{H}(2)$ commutes with γ^4 (i.e. $t\gamma^4 = \gamma^4 t$) iff $c = -b$, $d = a$, and therefore:

Corollary II.2.5.:

$$\mathrm{Spin}(1,3) = \mathrm{Spin}(1,4)_{\gamma^4} = \text{stabilizer of } \gamma^4$$

$$\mathrm{Spin}^+(1,3) = \mathrm{Spin}^+(1,4)_{\gamma^4} \ . \qquad \square$$

This corollary justifies (3.3) of Ch. I.

We shall now explicitly establish the isomorphisms:
$\overset{+}{\mathrm{R}}_{1,3} \cong \mathrm{C}(2)$, $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2;\mathrm{C})$.

Consider the mapping $\chi : \mathrm{H} \to \mathrm{C}(2)$ by

$$a = a^0 + \mathrm{i}a^1 + \mathrm{j}a^2 + \mathrm{k}a^3 \longrightarrow \begin{pmatrix} a^0 - \mathrm{i}a^3 & -a^2 - \mathrm{i}a^1 \\ a^2 - \mathrm{i}a^1 & a^0 + \mathrm{i}a^3 \end{pmatrix}$$

or $\chi(a) = a^0 1_2 - \mathrm{i}\, \vec{a}\cdot\vec{\sigma}$ where $\vec{a}\cdot\vec{\sigma} = a^1\sigma^1 + a^2\sigma^2 + a^3\sigma^3$ and $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

It is readily verified that χ is a real algebra isomorphism onto $\chi(\mathrm{H})$.

Now define:

$$\psi : \overset{+}{\mathrm{R}}_{1,3} \longrightarrow \mathrm{C}(2) \quad \text{by}$$

$$\psi\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \chi(a) + \mathrm{i}\ \chi(b)$$

One now checks that ψ is a real algebra isomorphism. Furthermore, an easy calculation reveals that $\det(\chi(a)+i\chi(b)) = (|a|^2-|b|^2) + i(\bar{a}b+\bar{b}a)$ and consequently we have:

<u>Theorem II.2.6.</u>:

$$\mathrm{Spin}(1,3) \cong \{t \in \mathbb{C}(2) : \det t = \pm 1\}$$

$$\mathrm{Spin}^+(1,3) \cong SL(2;\mathbb{C}) \qquad \square$$

It is interesting to note that if $\psi\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = t \in \mathbb{C}(2)$, then $\psi\left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}^*\right) = t^*$ where t^* is the adjoint of $t \in \mathbb{C}(2)$ associated with the correlated space $\mathbb{C}_{sp}$ (i.e. $\mathbb{C}_{sp} \equiv \mathbb{C}^{\psi}{}_{sp}$, where $\psi = \mathrm{id}_{\mathbb{C}}$; see (1.5) and (1.11)):

$$t = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} , \quad t^* = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} ; \quad \alpha,\beta,\gamma,\delta \in \mathbb{C} .$$

Things fall nicely into place because it's now clear that $t^*t = \det t \cdot 1_2$, thereby providing an alternate verification of Thm. II.2.6 (note that $\psi(\gamma^0\gamma^3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\psi(\gamma^1\gamma^3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\psi(\gamma^2\gamma^3) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ is an orthonormal generating set for $\overset{+}{\mathbb{R}}_{1,3} = \mathbb{R}_{1,2}$ which behaves correctly under the adjoint defined above).

<u>Galilei</u>

Thanks to the embeddings $\mathbb{R}^{1,0,3} \subset \mathbb{R}^{1,4}$ and $\mathbb{R}_{1,0,3} \subset \mathbb{R}_{1,4}$, we may use the results of the de Sitter case to obtain those in the Galilei situation.

Recalling (2.1), we have as an orthonormal basis of $\mathbb{R}^{1,0,3}$ which generates $\mathbb{R}_{1,0,3} = \langle\tilde{\Gamma}^0,\tilde{\Gamma}^1,\tilde{\Gamma}^2,\tilde{\Gamma}^3\rangle$:

$$\tilde{\Gamma}^0 = \Gamma^0 + \Gamma^4 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \oplus \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \tilde{\gamma}^0 \oplus -\tilde{\gamma}^0 \tag{2.3a}$$

$$\tilde{\Gamma}^1 = \Gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \tilde{\gamma}^1 \oplus -\tilde{\gamma}^1 \tag{2.3b}$$

$$\tilde{\Gamma}^2 = \Gamma^2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix} = \tilde{\gamma}^2 \oplus -\tilde{\gamma}^2 \tag{2.3c}$$

$$\tilde{\Gamma}^3 = \Gamma^3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix} = \tilde{\gamma}^3 \oplus -\tilde{\gamma}^3 \tag{2.3d}$$

As for $\overset{+}{\mathbb{R}}_{1,0,3}$, note that $\tilde{\Gamma}^i\tilde{\Gamma}^j = \tilde{\gamma}^i\tilde{\gamma}^j \oplus \tilde{\gamma}^i\tilde{\gamma}^j$ and consequently $\overset{+}{\mathbb{R}}_{1,0,3} \cong \langle\tilde{\gamma}^i\tilde{\gamma}^j : 0 \leq i < j \leq 3\rangle \subset \mathbb{H}(2)$.

From Cor. I.3.6, noting that $\tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3 = -\tilde{\gamma}^0$ and the trivial change of notation, we have:

<u>Theorem II.2.7.</u>:

$$\mathrm{Spin}(1,0,3) = \left\{ \begin{pmatrix} \alpha-\beta & -\beta \\ \beta & \alpha+\beta \end{pmatrix} \varepsilon\ \mathbb{H}(2) : |\alpha| = 1 \ , \ \bar{\alpha}\beta+\bar{\beta}\alpha = 0 \right\} \qquad \square$$

This is the Galileian analogue of Thm. II.2.4; there is also an analogue of Cor. II.2.5, namely:

<u>Corollary II.2.8.</u>:

$$\mathrm{Spin}(1,0,3) = \overset{+}{\mathrm{Spin}}(1,4)_{\gamma^0+\gamma^4} \ .$$

<u>Proof</u>: It suffices to show that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\ \overset{+}{\mathrm{Spin}}(1,4)$ commutes with $\gamma^0+\gamma^4 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ if and only if $a = \alpha-\beta$, $b = -\beta = -c$, $d = \alpha+2\beta$

where $|\alpha| = 1$, $\bar{\alpha}\beta+\bar{\beta}\alpha = 0$. This is straightforward using Thm. II.2.1.

□

By a similarity transformation, we obtain a neater form of Thm. II.2.7:

$$\mathrm{Spin}(1,0,3) = \left\{ \begin{pmatrix} \alpha & 0 \\ 2\beta & \alpha \end{pmatrix} \in H(2) : |\alpha| = 1 , \bar{\alpha}\beta+\bar{\beta}\alpha = 0 \right\}$$

In fact, $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ accomplishes this.

II.3. MATRIX LIE ALGEBRAS

The bases of the de Sitter, Lorentz, and homogeneous Galilei Lie algebras are listed (c.f. (4.12), (4.14), (4.15) of Ch. I). In keeping with Thm. II.2.1, II.2.4, II.2.7 the $H(2)$ representations, only, are provided.

de Sitter

We substitute (2.1) into (4.12) of Ch. I:

$$\underset{S}{J}^1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} ; \quad \underset{S}{J}^2 = \frac{1}{2}\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} ; \quad \underset{S}{J}^3 = \frac{1}{2}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$\underset{S}{K}^1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} ; \quad \underset{S}{K}^2 = \frac{1}{2}\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} ; \quad \underset{S}{K}^3 = \frac{1}{2}\begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$$

$$\underset{S}{P}^1 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} ; \quad \underset{S}{P}^2 = \frac{1}{2}\begin{pmatrix} -j & 0 \\ 0 & j \end{pmatrix} ; \quad \underset{S}{P}^3 = \frac{1}{2}\begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix}$$

$$\underset{S}{H} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Lorentz

If one uses (4.14) of Ch. I, then:

$$\underset{L}{J}^1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} ; \quad \underset{L}{J}^2 = \frac{1}{2}\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} ; \quad \underset{L}{J}^3 = \frac{1}{2}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$\underset{L}{K}^1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} ; \quad \underset{L}{K}^2 = \frac{1}{2}\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} ; \quad \underset{L}{K}^3 = \frac{1}{2}\begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$$

On the other hand, using (2.2) and the definitions: $\underset{L}{J}^A = \frac{1}{4}\epsilon^A{}_{BC}\gamma^B\gamma^C$, $\underset{L}{K}^A = \frac{1}{2}\gamma^0\gamma^A$, has the effect of replacing $\underset{L}{K}^A$ by $-\underset{L}{K}^A$ in the matrices above. Neither choice is to be preferred over the other.

Galilei

Combining (2.3) and (4.15) of Ch. I, one has:

$$\underset{G}{\mathcal{J}}^1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \; ; \; \underset{G}{\mathcal{J}}^2 = \frac{1}{2} \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \; ; \; \underset{G}{\mathcal{J}}^3 = \frac{1}{2} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$\underset{G}{K}^1 = \frac{1}{2} \begin{pmatrix} i & i \\ -i & -i \end{pmatrix} \; ; \; \underset{G}{K}^2 = \frac{1}{2} \begin{pmatrix} j & j \\ -j & -j \end{pmatrix} \; ; \; \underset{G}{K}^3 = \frac{1}{2} \begin{pmatrix} k & k \\ -k & -k \end{pmatrix}$$

In each of the three cases, the exponential map from the Lie algebra to its spin group is the ordinary matrix exponential.

II.4. FOUR-DIMENSIONAL COMPLEX REPRESENTATIONS

While we shall not undertake a classification of the complex representations of the Clifford algebras and spin groups of the last sections, it does, however, seem appropriate to offer an alternative to the quaternionic representations which have prevailed up to this point.

The following, in the non-degenerate case, is quite standard.

Lemma II.4.1.: If $\gamma^1, \gamma^2, \cdots, \gamma^n \in C(m)$ and form an orthonormal set of type (r,p,q) with $n = r+p+q$ and $p+q \geq 2$, then (i) m is even and (ii) $tr(\gamma^i) = 0$, $tr(\gamma^i\gamma^j) = 0$, $i \neq j$, (tr denotes trace).

Proof: (ii) As $p+q \geq 2$, at least two of $\{\gamma^i\}$ have non-vanishing squares, say γ^{n-1} and γ^n. For $i \neq n$, we have $\gamma^i = \pm \gamma^i(\gamma^n)^2 = \pm \gamma^i\gamma^n\gamma^n$, so that: $tr(\gamma^i) = \pm tr(\gamma^i\gamma^n\gamma^n) = \pm tr(\gamma^n\gamma^i\gamma^n) = \mp tr(\gamma^i\gamma^n\gamma^n) = - tr(\gamma^i)$ using $\gamma^i\gamma^n = -\gamma^n\gamma^i$ and properties of trace. Thus $tr(\gamma^i) = 0$. A similar argument involving $\gamma^n = \pm \gamma^n(\gamma^{n-1})^2$ implies that $tr(\gamma^n) = 0$. From $\gamma^i\gamma^j = -\gamma^j\gamma^i$, $i \neq j$, we have $tr(\gamma^i\gamma^j) = -tr(\gamma^j\gamma^i) = -tr(\gamma^i\gamma^j)$ and hence $tr(\gamma^i\gamma^j) = 0$.

(i) Since $(\gamma^n)^2 = \pm 1_m$, the eigenvalues of γ^n are 1, -1 or i,-i respectively; and because $tr(\gamma^n)$ = sum of eigenvalues of $\gamma^n = 0$, these $\pm$ signs occur equally often. Therefore, m is even.

□

The de Sitter situation is described in:

Proposition II.4.2.: Let $\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 \in C(m)$ be an orthonormal set of type $(1,4)$. Then $m \geq 4$; and when $m = 4$, $\langle\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4\rangle$ is

a non-universal Clifford algebra for $\mathbb{R}^{1,4}$. Moreover when $m = 4$, orthonormal sets of type (1,4) belong to one of two equivalence classes:

$$\text{either} \quad \{\tilde{\gamma}^0, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3, \tilde{\gamma}^4\}$$
$$\text{or} \quad \{\tilde{\gamma}^0, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3, -\tilde{\gamma}^4\}$$

where $\tilde{\gamma}^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$, $\tilde{\gamma}^A = \begin{pmatrix} 0 & \tau^A \\ \tau^A & 0 \end{pmatrix}$, $\tilde{\gamma}^4 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$, $1 \leq A \leq 3$,

and $\{i\, \tau^A = \sigma^A\}$ are the Pauli matrices.

<u>Proof</u>: From the remarks (esp. (iii)) immediately preceding Thm. I.1.3, we know that $\dim_{\mathbb{R}} \langle \gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 \rangle \geq 2^{5-1} = 16$; and since $\dim_{\mathbb{R}} \mathbb{C}(2) = 8 < 16$, we use Lemma II.4.1 to conclude that $m \geq 4$.

Suppose now that $m = 4$. Then $\langle \gamma^0, \gamma^1, \gamma^2, \gamma^3 \rangle$ generates $\mathbb{R}_{1,3}$ (again, see the remarks preceding Thm. I.1.3) and in addition, the γ^α 's , $0 \leq \alpha \leq 3$, are unique up to equivalence as members of $\mathbb{C}(4)$ (see Varadarajan (1970), Thm. 12.5). For the essentially unique $\tilde{\gamma}^\alpha$, $0 \leq \alpha \leq 3$, we use:

$$\tilde{\gamma}^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} , \quad \tilde{\gamma}^A = \begin{pmatrix} 0 & \tau^A \\ \tau^A & 0 \end{pmatrix} , \quad 1 \leq A \leq 3$$

(The motivation for this choice as follows. The mapping $\chi : \mathbb{H} \to \mathbb{C}(2)$, defined just prior to Thm. II.2.6, sends i to τ^1 , j to τ^2 , k to τ^3 and 1 to 1_2 ; it then induces a map $\mathbb{H}(2) \to \mathbb{C}(4)$ which sends $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ of (2.1) to the representatives given above). Thus, $\gamma^\alpha = U \tilde{\gamma}^\alpha U^{-1}$, $0 \leq \alpha \leq 3$, for some invertible $U \in \mathbb{C}(4)$.

Now, γ^4 is essentially determined up to a sign. For, $0 = \gamma^4\gamma^\alpha + \gamma^\alpha\gamma^4$ implies that $\gamma^4\gamma^\alpha\gamma^\beta = \gamma^\alpha\gamma^\beta\gamma^4$, $\alpha \neq \beta$, and so $\gamma^4 \in Z(\overset{+}{R}_{1,3})$. However $Z(\overset{+}{R}_{1,3}) = R\,\gamma^0\gamma^1\gamma^2\gamma^3$ (by Prop. I.1.7), so that $\gamma^4 = \pm\,\gamma^0\gamma^1\gamma^2\gamma^3$ and $U^{-1}\gamma^4 U = \pm\,\tilde\gamma^0\tilde\gamma^1\tilde\gamma^2\tilde\gamma^3$. But $\tilde\gamma^0\tilde\gamma^1\tilde\gamma^2\tilde\gamma^3 = -\tilde\gamma^4$, with the result that $\gamma^4 = \mp U\tilde\gamma^4 U^{-1}$. This completes the proof. $\square$

A faithful representation of $R_{1,4}$ is realizable within $C(8)$. In fact, since $R_{1,4} = {}^2H(2)$, and $H(2)$ sits inside $C(4)$, it is also true that $R_{1,4} \subset {}^2C(4) \subset C(8)$.

From the two inequivalent sets of γ^i's, $0 \leq i \leq 4$, as in Prop. II.4.2, we obtain two inequivalent sets $\{\tilde\gamma^0 \pm \tilde\gamma^4, \tilde\gamma^1, \tilde\gamma^2, \tilde\gamma^3\}$ of $C(4)$, orthonormal sets of type $(1,0,3)$. That this essentially tells the whole story, is the content of the next proposition.

<u>Proposition II.4.3.</u>: Let $\gamma^0, \gamma^1, \gamma^2, \gamma^3 \in C(m)$ be an orthonormal subset of $R^{1,0,3}$ (with $(\gamma^0)^2 = 0$). Then, $m \geq 4$; and when $m = 4$, $\langle\gamma^0, \gamma^1, \gamma^2, \gamma^3\rangle$ is a non-universal Clifford algebra for $R^{1,0,3}$. Further, $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ is equivalent either to $\{\tilde\gamma^0 + \tilde\gamma^4, \tilde\gamma^1, \tilde\gamma^2, \tilde\gamma^3\}$ or to $\{\tilde\gamma^0 - \tilde\gamma^4, \tilde\gamma^1, \tilde\gamma^2, \tilde\gamma^3\}$, the $\tilde\gamma^i$'s being those of the last proposition.

<u>Proof</u>: From Thm. I.1.3, $\dim_R\langle\gamma^0, \gamma^1, \gamma^2, \gamma^3\rangle \geq 12$; since $\dim_R C(2) = 8$, we must have $m \geq 4$ by Lemma II.4.1.

Suppose then that $m = 4$. By performing similarity transformations, one may verify that $\tilde\gamma^0 + \tilde\gamma^4$, $\tilde\gamma^A$ are similar to $\begin{pmatrix} 0 & 0 \\ 1_2 & 0 \end{pmatrix}$, $\begin{pmatrix} \tau^A & 0 \\ 0 & -\tau^A \end{pmatrix}$, for $1 \leq A \leq 3$, respectively and also that $\tilde\gamma^0 - \tilde\gamma^4$, $\tilde\gamma^A$ are similar to $\begin{pmatrix} 0 & 1_2 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} \tau^A & 0 \\ 0 & -\tau^A \end{pmatrix}$, for $1 \leq A \leq 3$, respectively. In

these forms, it is fairly easily verified that $\{\tilde\gamma^0+\tilde\gamma^4, \tilde\gamma^A : 1 \leq A \leq 3\}$ and $\{\tilde\gamma^0-\tilde\gamma^4, \tilde\gamma^A : 1 \leq A \leq 3\}$ are not equivalent. It remains to show that $\{\gamma^0,\gamma^1,\gamma^2,\gamma^3\}$ is equivalent to one of these.

First of all, $(\gamma^3)^2 = -1_4$ and $\mathrm{tr}(\gamma^3) \Rightarrow \gamma^3$ has eigenvalues $-i,-i,i,i$; likewise for $\gamma^1\gamma^2$. Furthermore, $\gamma^1\gamma^2$ commutes with γ^3 and since both $\gamma^1\gamma^2$ and γ^3 are diagonalizable (analyzing the Jordan canonical forms), they are simultaneously diagonalizable. Suppose then that γ^3 and $\gamma^1\gamma^2$ are similar to $\begin{pmatrix} \tau^3 & 0 \\ 0 & -\tau^3 \end{pmatrix}$ and $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ respectively (where $\lambda_j = \pm i$, $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0$ and $\tau^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$). Because $\lambda_1 = \lambda_4 \Rightarrow \lambda_2 = \lambda_3$ and therefore $\gamma^1\gamma^2 = \pm\gamma^3$, which in turn implies that $\gamma^0 = 0$ ($0 = \gamma^0\gamma^3+\gamma^3\gamma^0 = \pm\gamma^0\gamma^1\gamma^2 \pm\gamma^1\gamma^2\gamma^0 = \pm 2\,\gamma^0\gamma^1\gamma^2 = 2\,\gamma^0\gamma^3 \Rightarrow \gamma^0 = 0$), we must have instead: $\lambda_1 = -\lambda_4$, $\lambda_2 = -\lambda_3$. Then performing similarity transformations (which leave $\begin{pmatrix} \tau^3 & 0 \\ 0 & -\tau^3 \end{pmatrix}$ alone) to interchange λ_1,λ_4 and λ_2,λ_3 if necessary, we may suppose that $\lambda_1 = -\lambda_4 = -i$ and $\lambda_2 = -\lambda_3 = i$. All this proves the existence of an invertible U such that:

$$U^{-1}\gamma^3 U = \begin{pmatrix} \tau^3 & 0 \\ 0 & -\tau^3 \end{pmatrix} \quad , \quad U^{-1}\gamma^1\gamma^2 U = \begin{pmatrix} \tau^3 & 0 \\ 0 & \tau^3 \end{pmatrix} \quad .$$

By demanding $0 = \gamma^0\gamma^3 + \gamma^3\gamma^0$ and $0 = \gamma^0\gamma^1\gamma^2 - \gamma^1\gamma^2\gamma^0$, we find $U^{-1}\gamma^0 U = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $b\tau^3 = \tau^3 b$, $c\tau^3 = \tau^3 c$ and $bc=cb=0$ (as $(\gamma^0)^2 = 0$). Furthermore, $0 = \gamma^A\gamma^3 + \gamma^3\gamma^A$ for $A = 1,2$ and taking into account the form of $U^{-1}\gamma^1\gamma^2 U$ we have:

$$U^{-1}\gamma^1 U = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad , \quad U^{-1}\gamma^2 U = \begin{pmatrix} \tau^3 a & 0 \\ 0 & \tau^3 d \end{pmatrix}$$

with $a\tau^3 + \tau^3 a = 0 = d\tau^3 + \tau^3 d$, $a^2 = d^2 = -1_2$. Finally, $0 = \gamma^0\gamma^1 + \gamma^1\gamma^0$ implies $ab+bd = dc+ca = 0$.

The relations $b\tau^3 = \tau^3 b$, $c\tau^3 = \tau^3 c$, $bc=cb=0$ imply $b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$, $c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$, $b_1c_1 = b_2c_2 = 0$; $a\tau^3 + \tau^3 a = 0 = d\tau^3 + \tau^3 d$, $a^2 = d^2 = -1_2$ imply $a = \begin{pmatrix} 0 & a_1 \\ -a_1^{-1} & 0 \end{pmatrix}$, $d = \begin{pmatrix} 0 & d_1 \\ -d_1^{-1} & 0 \end{pmatrix}$.

Combining these in $ab+bd = 0 = dc+ca$ we find $b_2 = -b_1d_1a_1^{-1}$ and $c_2 = -c_1a_1d_1^{-1}$; as a result of $b_1c_1 = b_2c_2 = 0$ and $\gamma^0 \neq 0$, we have $b_1 = b_2 = 0$, c_1 and c_2 non-zero or $c_1 = c_2 = 0$, b_1 and b_2 non-zero.

Let

$$V = \begin{pmatrix} i & & & \\ & a_1^{-1} & & \\ & & -i & \\ & & & d_1^{-1} \end{pmatrix}$$

then

$$V^{-1}U^{-1}\gamma^3 UV = \begin{pmatrix} \tau^3 & 0 \\ 0 & -\tau^3 \end{pmatrix} \quad , \quad V^{-1}U^{-1}\gamma^1 UV = \begin{pmatrix} \tau^1 & 0 \\ 0 & -\tau^1 \end{pmatrix}$$

$$V^{-1}U^{-1}\gamma^2 UV = \begin{pmatrix} \tau^2 & 0 \\ 0 & -\tau^2 \end{pmatrix} \quad \text{and} \quad V^{-1}U^{-1}\gamma^0 UV = \begin{pmatrix} & & -b_1 & 0 \\ & & 0 & -b_1 \\ -c_1 & 0 & & \\ 0 & -c_1 & & \end{pmatrix}$$

with $b_1c_1 = 0$.

If $b_1 = 0 \neq c_1$ (resp. $c_1 = 0 \neq b_1$), there is a similarity transformation that leaves $V^{-1}U^{-1}\gamma^A UV$ alone, for $1 \leq A \leq 3$, and which sends $V^{-1}U^{-1}\gamma^0 UV$ to $\begin{pmatrix} 0 & 0 \\ 1_2 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & 1_2 \\ 0 & 0 \end{pmatrix}$).

Thus we have two possibilities (~ denotes similarity) for $\gamma^0, \gamma^1, \gamma^2, \gamma^3$:

(i) $\gamma^0 \sim \begin{pmatrix} 0 & 0 \\ 1_2 & 0 \end{pmatrix}$, $\gamma^A \sim \begin{pmatrix} \tau^A & 0 \\ 0 & -\tau^A \end{pmatrix}$ for $1 \leq A \leq 3$

(ii) $\gamma^0 \sim \begin{pmatrix} 0 & 1_2 \\ 0 & 0 \end{pmatrix}$, $\gamma^A \sim \begin{pmatrix} \tau^A & 0 \\ 0 & -\tau^A \end{pmatrix}$ for $1 \leq A \leq 3$.

This completes the proof. □

The proof of this last proposition is close in spirit to the analysis of the Lorentz case by Dumais (1977) to whom the present author is grateful for discussions on such matters.

The four dimensional complex representations of $\mathsf{R}_{1,4}$ (and hence also $\mathsf{R}_{1,3}$) and $\mathsf{R}_{1,0,3}$ provide complex analogues of Thm. II.2.1, 2.4 and 2.7; these complex forms of the spin groups are obtained through the mapping $\chi : \mathsf{H} \to C(2)$ as described preceding Thm. II.2.6.

<u>Corollary II.4.4.:</u>

$$\mathrm{Spin}(1,4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C(4) : a,b,c,d \in \chi(\mathsf{H}) \text{ and } a^*a - c^*c = d^*d - b^*b = \pm 1_2 \text{ , } a^*b = c^*d \right\}$$

where * denotes Hermitean conjugation on $C(2)$; an analogous statement holds for $\mathrm{Spin}^+(1,4)$. $\square$

Corollary II.4.5.:

$$\mathrm{Spin}(1,3) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \varepsilon\ C(4) : a,b \ \varepsilon\ \chi(H) \ ; \ a^*a-b^*b = \pm 1_2 \ , \ a^*b+b^*a=0 \right\}$$

and analogously for $\mathrm{Spin}^+(1,3)$. $\square$

Corollary II.4.6.:

$$\mathrm{Spin}(1,0,3) = \left\{ \begin{pmatrix} a-b & -b \\ b & a+b \end{pmatrix} \varepsilon\ C(4) : a,b \ \varepsilon\ \chi(H) \ ; \ a^*a = 1_2 \ , \ a^*b+b^*a = 0 \right\} \square$$

These corollaries serve to realize the Spin^+ groups as subgroups of $\mathrm{Spin}^+(2,4) \cong SU(2,2)$:

$$SU(2,2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\ C(4) : a^*a-c^*c = d^*d-b^*b = 1_2 \ , \ a^*b = c^*d \right\}$$

$$= \left\{ g \ \varepsilon\ C(4) : g^* \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix} g = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \right\} .$$

From Cor. II.4.5 we obtain the traditional "4-spinor" $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $SL(2;C)$. The details are as follows:

(i) conjugate $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ by $\begin{pmatrix} 1_2 & -i1_2 \\ 1_2 & i1_2 \end{pmatrix}$ to obtain $\begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix}$

(ii) as a,b range over $\chi(H)$ such that $a^*a-b^*b = 1_2$, $a^*b+b^*a = 0$ we have $a+ib \ \varepsilon\ SL(2;C)$ and $a-ib = (a+ib)^{-1*}$

(see remarks preceding Thm. II.2.6). Thus

$$\mathrm{Spin}^{+}(1,3) = \left\{ \begin{pmatrix} m & 0 \\ 0 & m^{-1*} \end{pmatrix} : m \in SL(2;C) \right\}$$

$$= D^{(1/2,0)} \oplus D^{(0,1/2)} \quad .$$

From Cor. II.4.6 we obtain the Lévy-Leblond (1967) representation $\Delta^{1/2}$ as follows:

(i) from the remarks following Cor. II.2.8 we have

$$\mathrm{Spin}(1,0,3) = \left\{ \begin{pmatrix} a & 0 \\ 2b & a \end{pmatrix} \in C(4) : a,b \in \chi(H) \ , \ a^{*}a = 1_2 \ , \ a^{*}b+b^{*}a = 0 \right\}$$

(ii) $a \in \chi(H)$ such that $a^{*}a = 1_2$ is an element of Spin(3)

(iii) $2b = 2ba^{*}a$ and $2ba^{*} = -\frac{\vec{v}\cdot\vec{\tau}}{2}$ for some $\vec{v} \in R^3$.

See also Steinwedel (1976, §9) for comments on $\Delta^{1/2}$, which is there denoted $D^{(1/2,0)} \oplus D^{(0,1/2)}$.

II.5. NOTES

§1.

When X is a right A - linear space, with $A = {}^sK$, $X\cdot(0,\cdots,1,\cdots,0) = \{x\cdot(0,\cdots,1,\cdots,0) : x \in X\} = \{x\cdot e_i : x \in X\}$, where $\{e_i = (0,\cdots,1,\cdots,0) : 1 \leq i \leq s\}$ is the set of *primitive orthogonal idempotents* of ${}^sK$. To explain: $e_i \in {}^sK$ has zeros in all positions except for the i th where it has a one and clearly $e_i e_j = 0$ for $i \neq j$ (orthogonality), $e_i^2 = e_i$ (idempotence) and e_i cannot be written as a non-trivial sum of idempotents of ${}^sK$ (primitivity). The assumption on X that all $X\cdot e_i$, $1 \leq i \leq s$, be isomorphic as K - linear spaces is equivalent to the requirement that X be a direct sum of s copies of some right K - linear space; if $X = A^m$, the direct sum decomposition has been outlined already in the remarks on spinor space. When X is left A - linear, we require all $e_i\cdot X$ to be isomorphic, etc.

Since all our right A - linear spaces will be of the form A^m (and our left A - linear spaces of the form $(A^m)^L = \{(a_1,\cdots,a_m) : a_i \in A\}$), linear mappings are representable as matrices with entries in A. As there is in general no way for scalar multiplication (right or left) on $L(X,Y)$ to interact with scalar multiplication on X and Y, $L(X,Y)$ is typically neither a right nor a left A - linear space. It is, however, linear over the centre of A which always contains ${}^sR$ when $A = {}^sK$; when $L(X,Y)$ is to be considered a linear space, it is in this sense of linearity over ${}^sR$.

For our purposes, we require the theory only for $A = K$ or ${}^2K$. The anti-automorphism ψ associated with a correlation becomes an anti-involution under certain circumstances in these two cases. First of all, assuming ξ is symmetric or skew, for $x,y \in X$ and $a \in A$ we have:

$$<x,y>_\xi \psi^2(a) = \pm\, \psi(<y,x>_\xi)\psi(\psi(a)) = \pm\, \psi(\psi(a)<y,x>_\xi)$$

$$= \pm\, \psi(<y\cdot a,x>_\xi) = <x,y\cdot a>_\xi = <x,y>_\xi a$$

It is tempting to cancel the $<x,y>_\xi$'s to conclude that $\psi^2(a) = a$, but care is needed. When $A = K$, it is enough to require in addition that ξ be non-zero in order that $\psi^2(a) = a$ (for there exist $x,y \in X$ such that $<x,y>_\xi \neq 0$ and since K is a division algebra, we may cancel). When $A = {}^2K$, we assume $\psi(e_1) = e_2$ (with $e_1 = (1,0)$, $e_2 = (0,1) \in {}^2K$ as described earlier), and moreover that ξ is non-degenerate. Then there exist $x,y \in X$ such that $<x,y>_\xi$ is invertible (Porteous (1969), Prop. 11.14), thus permitting cancellation to imply $\psi^2(a) = a$.

If these conditions are relaxed, then the result may be false. For example, when $A = {}^2H$, $X = A$ and $\psi(x,y) = (q\,\bar{x}\,\bar{q},\bar{y})$ (q being a unit quaternion ($|q|=1$) with non-real square), the following correlation:

$$<(x,y),(x',y')> = (0,\bar{y}y')$$

is symmetric but $\psi^2 \neq \mathrm{id}_A$. The problems are that $\psi(e_1) = e_1$, and $< , >$ is degenerate.

The condition $\psi(e_1) = e_2$ of the last paragraph is important. More generally, for $A = {}^sK$ with primitive idempotents e_i , $1 \leq i \leq s$, one says that an (anti-) automorphism ψ of A (which necessarily permutes the e_i 's) is *irreducible* if there is no proper subset of $\{e_1, e_2, \cdots, e_s\}$ mapped to itself by ψ ; when $s = 2$, ψ is irreducible iff $\psi(e_1) = e_2$, $\psi(e_2) = e_1$. An A^ψ - linear map is *irreducible* if ψ is irreducible. It is not difficult to see that ${}^sK$ has irreducible (anti-) involutions only if $s = 1$ or 2 .

Along with Prop. II.1.2, it is a fact that the adjoint induced by an irreducible symmetric or skew correlation is a real algebra anti-involution (Porteous (1969), Prop. 11.29; Thm. 11.32, for a proof of Prop. II.1.2).

Porteous (1969, Cor. 10.55) classifies the irreducible anti-involutions of ${}^sK$ and then obtains several classifications of irreducible symmetric or skew correlated spaces (1969, Thm. 11.41 and corollaries).

As further references to parts of the material in this section one should consult Bourbaki (1970) and Godement (1963).

§2.

An explicit proof that Thm. II.2.1, II.2.3 both characterize Spin(1,4) is easy. In fact, if N_- denotes the norm (Ch. I, §2) on H(2) defined by the correlation $\langle\ ,\ \rangle_-$, and N_{sp} that defined by $\langle\ ,\ \rangle_{sp}$, then $\forall\ t \in H(2)$:

$$N_-(CtC^{-1}) = N_{sp}(t) \quad \text{where} \quad C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix}$$

Therefore conjugation by C establishes the required isomorphism.

These two theorems seem to have been discussed first by Takahashi, R. (1963), and considered further in Ström (1970).

CHAPTER III

PHYSICAL APPLICATIONS: ALGEBRAIC

III.0. INTRODUCTION

Physically, there are good reasons for the appearance of the de Sitter group in discussions of the homogeneous Galilei group; these are discussed in section 1. In addition, a comparison between the Lorentz and homogeneous Galilei groups, which illustrates the $c \to \infty$ (speed of light $\to \infty$) limit of the Lorentz group, is provided. Various senses in which one group "approaches" another group (in the limit of some parameter) have been investigated in physics; the one most thoroughly studied is briefly recalled in section 2 in preparation for a new limiting procedure involving Clifford algebras. This new limiting procedure is introduced in section 3 and a discussion of the Lorentz $\longrightarrow$ Galilei limit undertaken. Section 4 contains further remarks and notes on the previous sections.

Of the results stated in this chapter, only those of section 3 are claimed to be new.

III.1. G_6 ACTION ON R^5 : PHYSICAL INTERPRETATION

As has already been emphasized on several occasions (e.g. Ch. I, Sec. 3; Ch. II, Sec. 2), the homogeneous Galilei group G_6 is a (stability) subgroup of the de Sitter group S_{10} and analogously for their spin groups (for notation, see Ch. I, Sec. 3).

Mathematically at least, this is a consequence of the inclusion, $R^{1,0,3} \subset R^{1,4}$, of orthogonal spaces. That G_6 is a subgroup of S_{10} is also not without a physical interpretation which we now describe (see also Pinski (1968), Niederer and O'Raifeartaigh (1974) and Steinwedel (1976)).

Consider the kinematical description of a massive, non-relativistic, internally structureless particle in empty space-time. If $(\vec{q},t)$ and $(\hat{\vec{q}},\hat{t})$ are the space-time coordinates of the particle with respect to two observers $\mathcal{O},\hat{\mathcal{O}}$ who are in uniform relative motion, then:

$$\hat{t} = t \tag{1.1a}$$

$$\hat{\vec{q}} = R\vec{q} + \vec{v}t \tag{1.1b}$$

for some $\vec{v} \in R^3$, $R \in SO(3)$. (It is assumed that the observers' clocks and coordinate systems coincide, up to a rotation in space, at the event $(t,\vec{q}) = (0,\vec{0}) = (\hat{t},\hat{\vec{q}})$.) If $(E,\vec{p},m)$ and $(\hat{E},\hat{\vec{p}},\hat{m})$ denote the energy (in this case kinetic), momentum and mass of the particle as seen by $\mathcal{O}$ and $\hat{\mathcal{O}}$ respectively then from (1.1) and from the primitive assumption of particle-mass invariance, we have:

$$\hat{E} = E + \vec{v}\cdot R\vec{p} + \frac{1}{2} m\, \vec{v}\cdot\vec{v} \tag{1.2a}$$

$$\hat{\vec{p}} = R\vec{p} + m\vec{v} \tag{1.2b}$$

$$\hat{m} = m \tag{1.2c}$$

The relations (1.2) define the transformation law of energy-momentum-mass between two Galileian observers; (1.2) remains valid if the particle possesses an internal energy which, however, has a value independent of the observer.

If $U : \mathrm{R}^5 \to \mathrm{R}$ denotes the quadratic form:

$$U(E,\vec{p},m) = \vec{p}\cdot\vec{p} - 2mE$$

notice that the internal energies as seen by O and $\hat{O}$ are $-\frac{1}{2m} U(E,\vec{p},m)$ and $-\frac{1}{2\hat{m}} U(\hat{E},\hat{\vec{p}},\hat{m})$ respectively. The observer-independence of internal energy is expressed as:

$$\frac{1}{2m} U(E,\vec{p},m) = \frac{1}{2\hat{m}} U(\hat{E},\hat{\vec{p}},\hat{m}) \tag{1.3}$$

From (1.2) we obviously have (1.3). Moreover, realizing that the form U, when diagonalized, is essentially diag(-1,1,1,1,1) and that invariance of m ($\hat{m}=m$) implies stability (upon right multiplication) of the row vector $(0\ \vec{0}\ 1)$, we may conclude that (1.2c) and (1.3) imply (1.2). Thus we have:

<u>Proposition III.1.1.</u>: The homogeneous Galilei group G_6 is isomorphic to the connected component containing the identity of the group of linear transformations on the space of $(E,\vec{p},m)$'s which leave invariant both m and $E - \frac{1}{2m}\vec{p}\cdot\vec{p}$ (mass and internal energy). □

This result, of course, is nothing other than the restatement in physical terms of (3.2) of Ch. I.

For comparison and in preparation for subsequent sections, the relativistic analogues of (1.1), (1.2) will now be indicated (and in such a form as to illustrate the $c \to \infty$ limit).

Consider a massive, relativistic, internally structureless particle in empty (Minkowski) space-time. The formulae relating the space-time coordinates of the particle with respect to two observers $O, \hat{O}$ (in uniform relative motion) analogous to (1.1) are:

$$\hat{t} = \gamma t + \gamma \frac{\vec{v} \cdot R\vec{q}}{c^2} \tag{1.4a}$$

$$\hat{\vec{q}} = R\vec{q} + \frac{\gamma - 1}{|\vec{v}|^2} (\vec{v} \cdot R\vec{q})\vec{v} + \gamma \vec{v} t \tag{1.4b}$$

where $\gamma = (1 - \frac{|\vec{v}|^2}{c^2})^{-1/2}$, $\vec{v} \in R^3$, $R \in SO(3)$ (of course subject to the conditions that $|\vec{v}|$, $\left|\frac{d\vec{q}}{dt}\right| < c$). If $m, \hat{m}$ are respectively the masses of the particle as seen by the two observers then $\vec{p} = m \frac{d\vec{q}}{dt}$, $\hat{\vec{p}} = \hat{m} \frac{d\hat{\vec{q}}}{d\hat{t}}$ are the corresponding momenta, and $E = mc^2$, $\hat{E} = \hat{m}c^2$ the corresponding energies. These quantities are related by:

$$\hat{E} = \gamma E + \gamma \, \vec{v} \cdot R\vec{p} \tag{1.5a}$$

$$\hat{\vec{p}} = R\vec{p} + \frac{\gamma - 1}{|\vec{v}|^2} (\vec{v} \cdot R\vec{p})\vec{v} + \gamma \frac{E}{c^2} \vec{v} \tag{1.5b}$$

$$\hat{m} = \gamma (1 + \frac{\vec{v} \cdot R\dot{\vec{q}}}{c^2}) m \tag{1.5c}$$

where $\dot{\vec{q}} = \frac{d\vec{q}}{dt}$.

Now the quantity $m(1 - \frac{|\dot{\vec{q}}|^2}{c^2})^{1/2}$ defined with reference to O is equal to the corresponding quantity $\hat{m}(1 - \frac{|\dot{\hat{\vec{q}}}|^2}{c^2})^{1/2}$ defined with reference to $\hat{O}$; it is therefore an invariant, called the rest mass, and is denoted by m_0 (it is also related to $E,\vec{p}$ by $E^2 = m_0^{\,2}c^4 + c^2\ |\vec{p}|^2$). The relativistic analogue of kinetic energy (as seen by O) is $E_k = E - m_0c^2$. In re-expressing (1.5) in terms of the new variables $(E_k,\vec{p},m_0)$ and $(\hat{E}_k,\hat{\vec{p}},\hat{m}_0)$ we have:

$$\hat{E}_k = \gamma E_k + \gamma\vec{v}\cdot R\vec{p} + (\gamma-1)c^2\ m_0 \tag{1.6a}$$

$$\hat{\vec{p}} = \gamma\ \frac{E_k}{c^2}\ \vec{v} + R\vec{p} + \frac{\gamma-1}{|\vec{v}|^2}\ (\vec{v}\cdot R\vec{p})\vec{v} + \gamma m_0\vec{v} \tag{1.6b}$$

$$\hat{m}_0 = m_0 \tag{1.6c}$$

Noticing that:

$$\lim_{c\to\infty} m = m_0 = \lim_{c\to\infty} \hat{m}$$

$$\lim_{c\to\infty} \vec{p} = m_0\ \dot{\vec{q}} \quad ; \quad \lim_{c\to\infty} \hat{\vec{p}} = m_0(R\dot{\vec{q}}+\vec{v})$$

$$\lim_{c\to\infty} E_k = \frac{1}{2}\ m_0\ \ |\dot{\vec{q}}|^2 \quad ; \quad \lim_{c\to\infty} \hat{E}_k = \frac{1}{2}\ m_0\ \ |R\dot{\vec{q}}+\vec{v}|^2$$

we see that (1.2) is precisely the limit of (1.6) as $c \to \infty$. Not only are the non-relativistic quantities the limits of the relativistic ones, but also the member of G_6 which defines (1.2) is the limit of the member of L_6 defining (1.6); in fact:

$$\lim_{c\to\infty}\begin{pmatrix} \gamma & \gamma\vec{v}^{\,t}R & (\gamma-1)c^2 \\ \gamma\frac{\vec{v}}{c^2} & R+\frac{\gamma-1}{|\vec{v}|^2}\vec{v}\vec{v}^{\,t}R & \gamma\vec{v} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \vec{v}^{\,t}R & \frac{1}{2}|\vec{v}|^2 \\ 0 & R & \vec{v} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.7)$$

Actually, this last statement requires comment because really we are taking a limit not within a fixed group but within a family of groups. To be precise, if $L_6^{\,c}$ denotes the group of all transformations of the $(E,\vec{p})$'s as in (1.5 a,b), then $L_6^{\,c} = SO^+(B_c)$ where B_c is the bilinear form $\mathrm{diag}(-c^{-2},1,1,1)$; therefore $L_6^{\,c} \cong SO^+(1,3) = L_6$ and letting $c \to \infty$, we see that $L_6^{\,c} \to SO^+(1,0,3) = G_6$. This means that $\lim_{c\to\infty} \Lambda_c(\vec{v},R) = \begin{pmatrix} 1 & \vec{v}^{\,t}R \\ 0 & R \end{pmatrix}$, where $\Lambda_c(\vec{v},R) = \begin{pmatrix} \gamma & \gamma\vec{v}^{\,t}R \\ \gamma\frac{\vec{v}}{c^2} & R+\frac{\gamma-1}{|\vec{v}|^2}\vec{v}\vec{v}^{\,t}R \end{pmatrix}$.

In comparing (1.2) and (1.6), notice that the right-hand side of (1.7) leaves invariant the bilinear form whose matrix representative is $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1_3 & 0 \\ -1 & 0 & 0 \end{pmatrix}$; whereas the matrix under the limit symbol on the left-hand side leaves invariant the bilinear form represented by $\begin{pmatrix} -\frac{1}{c^2} & 0 & -1 \\ 0 & 1_3 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. These bilinear forms are equivalent to $\mathrm{diag}(-1,1,1,1,1)$ and $\mathrm{diag}(\lambda_-(c),1,1,1,\lambda_+(c))$ respectively (where $\lambda_\pm(c) = -\frac{1}{2}c^{-2} \pm \sqrt{1+\frac{1}{4}c^{-4}}$) and consequently G_6 and $L_6^{\,c}$ are realizable as subgroups of $SO^+(-1,1,1,1,1) = SO^+(1,4)$ and $SO^+(\lambda_-(c),1,1,1,\lambda_+(c)) \cong SO^+(1,4)$ respectively; the limit (1.7) is therefore a result of the limit $SO^+(\lambda_-(c),1,1,1,\lambda_+(c)) \to SO^+(1,4)$ as $c \to \infty$.

III.2. LIMITS OF LIE ALGEBRAS

In the last section, the homogeneous Galilei group (the kinematical group of non-relativistic physics) was shown to be a limit of the Lorentz group (the kinematical group of relativistic physics) as the "speed of light" was allowed to become infinite. Moreover, the physical quantities on which these groups act as transformation groups maintain their respective physical meanings in the limit $c \to \infty$. This of course has great intuitive appeal. However, upon further scrutiny, the calculations of section 1 are seen to be rather ad hoc in the sense that one must have a very clear interpretation of the variables on which the groups act; thus the remarks of section 1 belong more properly to a discussion of limits of transformation groups rather than simply of groups themselves. Although of undeniable importance, it is not our intention to enter into the issues of deformation theory (see, for example, Guillemin and Sternberg (1966)) and instead follow a simpler development.

To simplify matters further, the discussion will centre on limits of Lie algebras rather than of Lie groups, thereby side-stepping most topological questions. The fruitfulness of studying such limits was suggested by Segal (1951) in relation to problems of quantum field theory (see also Segal (1963), Ch. VIII); work was begun by Inönü and Wigner (1953) and further extended by Saletan (1961) and others. The techniques of Inönü, Wigner and Saletan (IWS), in slightly modified form, will be reviewed by way of example: namely, the homogeneous Galilei Lie algebra as a limit of the Lorentz Lie algebra.

As originally proposed by Segal (1951), one ought to regard a limit of a family of Lie algebras as the Lie algebra defined by the limiting structure constants. That is, suppose $\{\ell^\varepsilon : 0 < \varepsilon \leq 1\}$ is a family of isomorphic Lie algebras: $\ell^\varepsilon = (V, [\ ,\ ]_\varepsilon)$, where V is the underlying vector space of ℓ^ε and $[\ ,\ ]_\varepsilon$ the Lie bracket on V defining ℓ^ε. If $\{E_i\}$ is a basis of V, then the structure constants $\{C_{ij}{}^k(\varepsilon)\}$ of ℓ^ε, with respect to this basis, are defined by (summation convention!):

$$[E_i, E_j]_\varepsilon = C_{ij}{}^k(\varepsilon)\, E_k \tag{2.1}$$

If it happens that $\lim_{\varepsilon\to 0} C_{ij}{}^k(\varepsilon)$ exists for all i,j,k, then $\{C_{ij}{}^k(0) = \lim_{\varepsilon\to 0} C_{ij}{}^k(\varepsilon)\}$ is a set of structure constants for a Lie algebra, denoted by $\ell^0 = (V, [\ ,\ ]_0)$ where $[E_i, E_j]_0 = C_{ij}{}^k(0) E_k$. (In (2.1) we avoid topological questions by insisting V be finite dimensional). One says that ℓ^0 is the *limit* of the ℓ^ε's and writes $\ell^0 = \lim_{\varepsilon\to 0} \ell^\varepsilon$.

This situation may be recast in a slightly different form (see Saletan (1961)). Since all the ℓ^ε, for $\varepsilon > 0$, are assumed isomorphic, let $W_\varepsilon : \ell^\varepsilon \to \ell = \ell^1$ be an isomorphism for all $\varepsilon \in (0,1]$ (for notational simplicity write: $\ell^1 = \ell$, $[\ ,\ ]_1 = [\ ,\ ]$ and $C_{ij}{}^k(1) = C_{ij}{}^k$). Suppose that $W_\varepsilon(E_i) = E_j (W_\varepsilon)^j{}_i$, again using summation convention here (as well as throughout the rest of this chapter). The condition that W_ε be an isomorphism is:

$$W_\varepsilon[x,y]_\varepsilon = [W_\varepsilon x, W_\varepsilon y] \tag{2.2a}$$

for all $x,y \in V$ in addition to W_ε being non-singular. In terms of

the structure constants:

$$C_{ij}^{\ \ k}(\varepsilon) = (W_\varepsilon^{-1})^k_{\ n}\, C_{\ell m}^{\ \ n}(W_\varepsilon)^\ell_{\ i}(W_\varepsilon)^m_{\ j} \tag{2.2b}$$

Clearly, $\lim_{\varepsilon\to 0} C_{ij}^{\ \ k}(\varepsilon)$ exists for all i,j,k if and only if $\lim_{\varepsilon\to 0} [\cdot,\cdot]_\varepsilon = \lim_{\varepsilon\to 0} W_\varepsilon^{-1}[W_\varepsilon(\cdot),W_\varepsilon(\cdot)]$ is a viable Lie bracket on V. This need not be the case. In fact, if $\lim_{\varepsilon\to 0} W_\varepsilon$ is non-singular then ℓ^0 will be isomorphic to ℓ but if $\lim_{\varepsilon\to 0} W_\varepsilon$ is singular then ℓ^0, providing it exists, will not be isomorphic to ℓ.

This slightly altered point of view which emphasizes W_ε, allows a reformulation. Given a Lie algebra $\ell = (V,[\ ,\ ])$ and a family $\{W_\varepsilon : 0 < \varepsilon \leq 1,\ W_1 = id_V\}$ of isomorphisms of V (as a vector space not as a Lie algebra) depending continuously on ε, define a new Lie algebra $\ell^\varepsilon = (V,[\ ,\ ]_\varepsilon)$, where $[x,y]_\varepsilon = W_\varepsilon^{-1}[W_\varepsilon x,W_\varepsilon y]$. Clearly ℓ^ε and ℓ are isomorphic but the existence of $\lim_{\varepsilon\to 0}\ell^\varepsilon$ is the interesting question, particularly when $\lim_{\varepsilon\to 0} W_\varepsilon$ is singular. If V has a direct sum decomposiition $V = V_0 \oplus V_1$, $V_1 \neq (0)$ with respect to which $W_\varepsilon|V_0 = id_{V_0}$ and $W_\varepsilon|V_1 = \varepsilon\cdot id_{V_1}$ (i.e. if $W_\varepsilon = \begin{pmatrix} 1_{V_0} & 0 \\ 0 & \varepsilon\cdot 1_{V_1}\end{pmatrix}$) then $\lim_{\varepsilon\to 0}\ell$ if it exists is called an *IWS contraction* of ℓ. A necessary and sufficient condition that $\lim_{\varepsilon\to 0}\ell$ exist is given next (see Saletan (1961)).

<u>Proposition III.2.1.</u>: Given $V = V_0 \oplus V_1$ and $W_\varepsilon = id_{V_0} \oplus \varepsilon\cdot id_{V_1}$, $\lim_{\varepsilon\to 0}\ell^\varepsilon$ exists if and only if $(V_0,[\ ,\ ])$ is a subalgebra of

$\ell = (V, [\ ,\ ])$.

Proof: If $z = z^0 + z^1$ denotes the decomposition of a vector $z \in V$ with $z^0 \in V_0$, $z^1 \in V_1$ then:

$$[x,y]_\varepsilon = \varepsilon^{-1} \cdot [x^0,y^0]^1 + ([x^0,y^0]^0 + [x^0,y^1]^1 + [x^1,y^0]^1)$$

$$+ \varepsilon([x^0,y^1]^0 + [x^1,y^0]^0 + [x^1,y^1]^1) + \varepsilon^2 \cdot [x^1,y^1]^0$$

so $\lim_{\varepsilon \to 0} [x,y]_\varepsilon$ exists for all $x,y \in V$ if and only if $[x^0,y^0]^1 = 0$ for all $x,y \in V$ i.e. if and only if $[V_0,V_0] \subset V_0$. $\square$

Now, to the example.

The Lorentz Lie algebra ℓ_6 has a basis $\{J^A, K^A\}$ (see Ch. I, (4.14)) with Lie products:

$$[J^A, J^B] = \epsilon^{AB}{}_C\, J^C \tag{2.3a}$$

$$[J^A, K^B] = \epsilon^{AB}{}_C\, K^C \tag{2.3b}$$

$$[K^A, K^B] = -\epsilon^{AB}{}_C\, J^C \tag{2.3c}$$

We now must choose a subalgebra $(V_0, [\ ,\ ])$ of ℓ_6 , and because it is usually desirable for the contracted algebra ℓ_6^0 to contain $so(3)$ we shall take $V_0 \supset so(3)$ (see Bacry and Lévy-Leblond (1968) for a detailed discussion of this point). Now it is an easy calculation to verify that the only subalgebras of ℓ_6 containing $so(3) = \langle J^1, J^2, J^3 \rangle$ are $so(3)$ and ℓ_6 itself; thus the only non-trivial rotation-invariant contraction of ℓ_6 is that obtained by taking $V_0 = so(3)$. Thus, computing $[\ ,\ ]_\varepsilon$ (we choose $V_1 = \mathrm{span}_R\{K^1, K^2, K^3\}$):

$$[J^A, J^B]_\varepsilon = W_\varepsilon^{-1}[W_\varepsilon J^A, W_\varepsilon J^B] = W_\varepsilon^{-1}[J^A, J^B] = \epsilon^{AB}{}_C\, J^C \tag{2.4a}$$

$$[J^A, K^B]_\varepsilon = W_\varepsilon^{-1}[J^A, \varepsilon\cdot K^B] = \varepsilon\cdot W_\varepsilon^{-1}(\epsilon^{AB}{}_C K^C) = \epsilon^{AB}{}_C\, K^C \tag{2.4b}$$

$$[K^A, K^B]_\varepsilon = W_\varepsilon^{-1}[\varepsilon\cdot K^A, \varepsilon\cdot K^B] = \varepsilon^2 W_\varepsilon^{-1}(-\epsilon^{AB}{}_C J^C) = -\varepsilon^2 \epsilon^{AB}{}_C\, K^C \tag{2.4c}$$

and taking $\varepsilon \to 0$ we find:

$$[J^A, J^B]_0 = \epsilon^{AB}{}_C\, J^C \tag{2.5a}$$

$$[J^A, K^B]_0 = \epsilon^{AB}{}_C\, K^C \tag{2.5b}$$

$$[K^A, K^B]_0 = 0 \tag{2.5c}$$

Recalling the Lie relations (Ch. I, (4.16)) of the homogeneous Galilei Lie algebra g_6, we see that $\lim_{\varepsilon\to 0} \ell_6^{\,\varepsilon} = g_6$.

A similar statement may be made in the case of the Poincaré Lie algebra ℓ_{10}. If ℓ_{10} has a basis $\{J^A, K^A, P^A, H\}$ satisfying the appropriate Lie relations (see Ch. I, (6.7)), define for $\varepsilon > 0$:

$$W_\varepsilon(J^A) = J^A \;,\; W_\varepsilon(K^A) = \varepsilon\cdot K^A \;,\; W_\varepsilon(P^A) = \varepsilon\cdot P^A \;,\; W_\varepsilon(H) = H \;.$$

It is easy to show that the limiting Lie relations are those of g_{10}, the Lie algebra of the Galilei group G_{10} (see Ch. I, (6.3 a,b,c) for $m = 0$).

For further remarks on contractions refer to the Notes section.

III.3. CONTRACTION OF CLIFFORD ALGEBRAS

In this section, a notion of limit or contraction of a Clifford algebra will be defined in analogy with that of Lie algebra contraction. While the principal application shows that the homogeneous Galilei Clifford algebra $R_{1,0,3}$ is a limit of the Lorentz Clifford algebra $R_{1,3}$, a general result analogous to Prop. III.2.1 is proven and some of the relationships between Lie and Clifford algebra contractions pointed out.

Suppose C is a universal Clifford algebra for some real orthogonal space X of dimension n ; then C will be of dimension 2^n as will V , the underlying vector space of C . Let us suppose also that a family $\{W_\varepsilon : 0 < \varepsilon \leq 1 , W_1 = \mathrm{id}_V\}$ of isomorphisms of V , depending continuously on ε , is given. Define a new multiplication $\underset{\varepsilon}{*}$ on V for each $\varepsilon \in (0,1]$ as follows:

$$a \underset{\varepsilon}{*} b = W_\varepsilon^{-1}(W_\varepsilon(a)W_\varepsilon(b)) \tag{3.1}$$

for $a,b \in V$; and denote the algebra $(V,\underset{\varepsilon}{*})$ by C^ε . This notation anticipates the fact that C^ε is a Clifford algebra isomorphic to C . In fact, the multiplication $\underset{\varepsilon}{*}$ is associative (for if $a,b,c \in V$, then $a \underset{\varepsilon}{*} (b\underset{\varepsilon}{*}c) = W_\varepsilon^{-1}(W_\varepsilon(a)W_\varepsilon(b\underset{\varepsilon}{*}c)) = W_\varepsilon^{-1}(W_\varepsilon(a)W_\varepsilon(b)W_\varepsilon(c))$, by associativity of multiplication in C , and consequently $a \underset{\varepsilon}{*} (b\underset{\varepsilon}{*}c) = (a\underset{\varepsilon}{*}b) \underset{\varepsilon}{*} c$ is written $a\underset{\varepsilon}{*}b\underset{\varepsilon}{*}c$); moreover W_ε provides the algebra isomorphism (the identity in C^ε is $W_\varepsilon^{-1}(1)$ and trivially, $W_\varepsilon(a\underset{\varepsilon}{*}b) = W_\varepsilon(a)W_\varepsilon(b)$ almost by definition; if $\{\gamma^i\}$ is an orthonormal basis of X generating C with $\gamma^i\gamma^j + \gamma^j\gamma^i = -2B^{ij}$, then likewise will $\{W_\varepsilon^{-1}(\gamma^i)\}$ generate

$C^\varepsilon : W_\varepsilon^{-1}(\gamma^i) \underset{\varepsilon}{*} W_\varepsilon^{-1}(\gamma^j) + W_\varepsilon^{-1}(\gamma^j) \underset{\varepsilon}{*} W_\varepsilon^{-1}(\gamma^i) = -2B^{ij}\ W_\varepsilon^{-1}(1)$).

If $\lim_{\varepsilon\to 0} a \underset{\varepsilon}{*} b$ exists for all $a,b \in V$ then we write $C^0 = \lim_{\varepsilon\to 0} C^\varepsilon$ and call C^0 the *contraction* of $\{C^\varepsilon\}$.

Proposition III.3.1.: Suppose $V = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ and $W_\varepsilon = \mathrm{id}_{V_0} \oplus \varepsilon\cdot\mathrm{id}_{V_1} \oplus \cdots \oplus \varepsilon^k\cdot\mathrm{id}_{V_k}$. Then $\lim_{\varepsilon\to 0} a \underset{\varepsilon}{*} b$ exists for all $a,b \in V$ if and only if $V_iV_j \subset V_0 \oplus V_1 \oplus \cdots \oplus V_{i+j}$ for all i,j such that $i+j < k$.

Proof: Let $c = c_0 + c_1 + \cdots + c_k$ denote the decomposition of c with $c_i \in V_i$ so that $W_\varepsilon(c) = c_0 + \varepsilon\cdot c_1 + \cdots + \varepsilon^k\cdot c_k$ and $W_\varepsilon^{-1}(c) = c_0 + \varepsilon^{-1}\cdot c_1 + \cdots + \varepsilon^{-k}\cdot c_k$, etc. Clearly, $\lim_{\varepsilon\to 0} a \underset{\varepsilon}{*} b$ exists for all $a,b \in V$ if and only if $\lim_{\varepsilon\to 0} a_i \underset{\varepsilon}{*} b_j$ exists for all $i,j \in \{0,1,\cdots,k\}$ and all $a_i \in V_i$, $b_j \in V_j$. Given $a_i \in V_i$ and $b_j \in V_j$, we compute:

$$a_i \underset{\varepsilon}{*} b_j = W_\varepsilon^{-1}(W_\varepsilon(a_i)W_\varepsilon(b_j)) = W_\varepsilon^{-1}(\varepsilon^i\cdot a_i\cdot\varepsilon^j\cdot b_j) = \varepsilon^{i+j}\cdot W_\varepsilon^{-1}(a_ib_j)$$

$$= \varepsilon^{i+j}\cdot(a_ib_j)_0 + \varepsilon^{i+j-1}\cdot(a_ib_j)_1 + \cdots + (a_ib_j)_{i+j} + \cdots +$$

$$\varepsilon^{i+j-m}\cdot(a_ib_j)_m$$

and so $\lim_{\varepsilon\to 0} a_i \underset{\varepsilon}{*} b_j$ exists (and equals $(a_ib_j)_{i+j}$) if and only if $(a_ib_j)_k = 0$ whenever $k > i+j$. This can be restated as $V_iV_j \subset V_0 \oplus V_1 \oplus \cdots \oplus V_{i+j}$ which finishes the proof. □

The general contractions of $\mathcal{R}_{r,p,q}$ assuming particularly simple forms for the W_ε's will now be derived; this general result will be applied to the contraction $\mathcal{R}_{1,3} \to \mathcal{R}_{1,0,3}$.

Suppose $\{\gamma^i\}$ is an orthonormal basis of $R^{r,p,q}$ generating $R_{r,p,q}$. The mapping W_ε is to multiply selected γ^i's by ε and the rest by 1 and then be extended to all of $R_{r,p,q}$ in the more-or-less simplest way. Precisely:

$$W_\varepsilon(1) = 1 \tag{3.2a}$$

$$W_\varepsilon(\gamma^i) = \varepsilon^{\lambda(i)}\cdot\gamma^i \quad \text{where} \quad \lambda(i) \in \{0,1\} \tag{3.2b}$$

$$W_\varepsilon(\gamma^{i_1}\cdots\gamma^{i_m}) = W_\varepsilon(\gamma^{i_1})\cdots W_\varepsilon(\gamma^{i_m}) \quad \text{for} \quad 1 \leq i_1 < \cdots < i_m \leq n \tag{3.2c}$$

where $n = r+p+q$. This may be stated more concisely in the following manner:

$$W_\varepsilon(\gamma^I) = \varepsilon^{\lambda(I)}\cdot\gamma^I \tag{3.3}$$

where $I \subset \{1,2,\cdots,n\}$, $\lambda(I) = \sum_{i\varepsilon I} \lambda(i)$ and $\lambda(\phi) = 0$ (of course, I is not to have repeated members). Rather than attempt to compute the direct sum decomposition of V and verify the hypotheses of Prop. III. 3.1, we proceed directly to show that $\lim_{\varepsilon\to 0} R^\varepsilon_{r,p,q}$ exists. This will be true if we show that $\lim_{\varepsilon\to 0} \gamma^I \underset{\varepsilon}{*} \gamma^J$ exists for all $I,J \subset \{1,2,\cdots,n\}$.

Recalling the meaning of $\gamma^I\gamma^J = \mu(I,J)\gamma^{I\Delta J}$ (see the proof of Thm. I.1.3 and the notes on section 1 of Ch. I; in addition to $\mu(I,J) = \pm 1$ it may happen that $\mu(I,J) = 0$ if $\exists\, i \in I\cap J$ such that $(\gamma^i)^2 = 0$; as before $I\Delta J = (I\backslash J)\cup(J\backslash I)$), we may now compute $\gamma^I \underset{\varepsilon}{*} \gamma^J$ as follows:

$$\gamma^I \underset{\varepsilon}{*} \gamma^J = W_\varepsilon^{-1}(W_\varepsilon(\gamma^I)W_\varepsilon(\gamma^J)) = W_\varepsilon^{-1}(\varepsilon^{\lambda(I)+\lambda(J)}\gamma^I\gamma^J)$$

$$= \mu(I,J)\ \varepsilon^{\lambda(I)+\lambda(J)}\ W_\varepsilon^{-1}(\gamma^{I\Delta J})$$

$$= \mu(I,J)\ \varepsilon^{\lambda(I)+\lambda(J)-\lambda(I\Delta J)}\ \gamma^{I\Delta J}$$

$$= \varepsilon^{\lambda(I)+\lambda(J)-\lambda(I\Delta J)}\ \gamma^I\gamma^J \ .$$

However, λ is monotone increasing ($K \subset L \Rightarrow \lambda(K) \leq \lambda(L)$) and additive one disjoint unions ($K \cap L = \phi \Rightarrow \lambda(K \cup L) = \lambda(K) + \lambda(L)$) and since $I\Delta J = (I\backslash J) \cup (J\backslash I)$ is such a disjoint union with $(I\backslash J) \subset I$ and $(J\backslash I) \subset J$, we may conclude that $\lambda(I)+\lambda(J) - \lambda(I\Delta J) \geq 0$ so that $\lim_{\varepsilon\to 0} \varepsilon^{\lambda(I)+\lambda(J)-\lambda(I\Delta J)}$ exists implying the existence of $\lim_{\varepsilon\to 0} \gamma^I \underset{\varepsilon}{*} \gamma^J$ for arbitrary $I,J \subset \{1,2,\cdots,n\}$.

<u>Theorem III.3.2.</u>: If $W_\varepsilon(\gamma^I) = \varepsilon^{\lambda(I)}\cdot\gamma^I$ as in (3.3), then $\lim_{\varepsilon\to 0} R^\varepsilon_{r,p,q} = R_{r',p',q'}$ where $p-p'$ is the number of γ^i 's with $(\gamma^i)^2 = 1$ and $\lambda(i) = 1$, $q-q'$ is the number of γ^i 's with $(\gamma^i)^2 = -1$ and $\lambda(i) = 1$, and $r'-r = (p-p')+(q-q')$.

<u>Proof</u>: With this particular choice of W_ε 's, we have $\gamma^{i_1} \underset{\varepsilon}{*} \cdots \underset{\varepsilon}{*} \gamma^{i_m} = \gamma^{i_1}\cdots\gamma^{i_m}$ for $1 \leq i_1 < \cdots < i_m \leq n$ ($\gamma^{i_1} \underset{\varepsilon}{*} \cdots \underset{\varepsilon}{*} \gamma^{i_m} = W_\varepsilon^{-1}(W_\varepsilon(\gamma^{i_1}) \cdots W_\varepsilon(\gamma^{i_m})) = W_\varepsilon^{-1}(W_\varepsilon(\gamma^{i_1}\cdots\gamma^{i_m})) = \gamma^{i_1}\cdots\gamma^{i_m}$, since $i_1 < \cdots < i_m$, using the definition of W_ε). Moreover, if $i \neq j$ then $\gamma^i \underset{\varepsilon}{*} \gamma^j + \gamma^j \underset{\varepsilon}{*} \gamma^i = \gamma^i\gamma^j + \gamma^j\gamma^i = -2B^{ij} = 0$, and $\gamma^i \underset{\varepsilon}{*} \gamma^i = W_\varepsilon^{-1}(W_\varepsilon(\gamma^i)W_\varepsilon(\gamma^i)) = W_\varepsilon^{-1}(\varepsilon^{2\cdot\lambda(i)}(\gamma^i)^2) = W_\varepsilon^{-1}(-\varepsilon^{2\cdot\lambda(i)}B^{ii}) = -\varepsilon^{2\cdot\lambda(i)}B^{ii}$. Thus $\{\gamma^i\}$ is an orthonormal basis of $R^\varepsilon_{r,p,q} = (R^n, B(\varepsilon))$ where $B(\varepsilon)$ is the bilinear form: $B^{ij}(\varepsilon) = \varepsilon^{2\cdot\lambda(i)}B^{ij}$. A count of those entries

of $B(\varepsilon)$ which become zero in the limit $\varepsilon \to 0$ gives the stated result. □

Corollary III.3.3.: If $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ is an orthonormal basis of $\mathbb{R}^{1,3}$ generating $\mathbb{R}_{1,3}$ ($(\gamma^0)^2 = 1$, $(\gamma^A)^2 = -1$, etc.) and W_ε is of the form given in Prop. III.3.1 with

$$V_0 = \mathrm{span}_{\mathbb{R}}\{1, \gamma^A, \gamma^A\gamma^B, \gamma^1\gamma^2\gamma^3 : 1 \leq A \neq B \leq 3\} \quad \text{and}$$

$$V_1 = \mathrm{span}_{\mathbb{R}}\{\gamma^0, \gamma^0\gamma^A, \gamma^0\gamma^A\gamma^B, \gamma^0\gamma^1\gamma^2\gamma^3 : 1 \leq A \neq B \leq 3\} ,$$

then $\lim_{\varepsilon\to 0} \mathbb{R}^{\varepsilon}_{1,3} = \mathbb{R}_{1,0,3}$ (the homogeneous Galilei Clifford algebra is a contraction of the Lorentz Clifford algebra).

Proof: This is the statement of Thm. III.3.2 for $r = 0$, $p = 1$, $q = 3$ and $\lambda(0) = 1$, $\lambda(A) = 0$ for $1 \leq A \leq 3$. □

This section will close with a discussion, by example, of the relationship between Lie and Clifford algebra contraction. Specifically, it will be seen how the Clifford algebra contraction $\mathbb{R}_{1,3} \to \mathbb{R}_{1,0,3}$ induces the Lie algebra contraction $\ell_6 \to g_6$.

For a Lie algebra basis of ℓ_6 we use:

$$J^A = \frac{1}{4} \epsilon^A{}_{BC} \gamma^B\gamma^C \tag{3.4a}$$

$$K^A = \frac{1}{2} \gamma^0\gamma^A \tag{3.4b}$$

as indicated previously (Ch. I, (4.14)), where $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ form an orthonormal basis of $\mathbb{R}^{1,3}$ generating $\mathbb{R}_{1,3}$. Following the scheme of Cor. III.3.3 we calculate:

$$J^A \underset{\varepsilon}{*} J^B - J^B \underset{\varepsilon}{*} J^A = W_\varepsilon^{-1}(W_\varepsilon(J^A)W_\varepsilon(J^B) - W_\varepsilon(J^B)W_\varepsilon(J^A))$$

$$= W_\varepsilon^{-1}(J^A J^B - J^B J^A) = W_\varepsilon^{-1}(\epsilon^{AB}{}_C J^C)$$

$$= \epsilon^{AB}{}_C\, J^C = [J^A, J^B] = [J^A, J^B]_\varepsilon \qquad (3.5a)$$

$$J^A \underset{\varepsilon}{*} K^B - K^B \underset{\varepsilon}{*} J^A = W_\varepsilon^{-1}(W_\varepsilon(J^A)W_\varepsilon(K^B) - W_\varepsilon(K^B)W_\varepsilon(J^A))$$

$$= W_\varepsilon^{-1}(J^A \cdot \varepsilon K^B - \varepsilon K^B \cdot J^A) = W_\varepsilon^{-1}(\varepsilon\ \epsilon^{AB}{}_C\, K^C)$$

$$= \epsilon^{AB}{}_C\, K^C = [J^A, K^B] = [J^A, K^B]_\varepsilon \qquad (3.5b)$$

$$K^A \underset{\varepsilon}{*} K^B - K^B \underset{\varepsilon}{*} K^A = W_\varepsilon^{-1}(W_\varepsilon(K^A)W_\varepsilon(K^B) - W_\varepsilon(K^B)W_\varepsilon(K^A))$$

$$= W_\varepsilon^{-1}(\varepsilon K^A \cdot \varepsilon K^B - \varepsilon K^B \cdot \varepsilon K^A) = W_\varepsilon^{-1}(-\varepsilon^2\ \epsilon^{AB}{}_C\, J^C)$$

$$= -\varepsilon^2\ \epsilon^{AB}{}_C\, J^C = \varepsilon^2 [K^A, K^B] = [K^A, K^B]_\varepsilon \qquad (3.5c)$$

where use has been made of the relations $W_\varepsilon(J^A) = J^A$, $W_\varepsilon(K^A) = \varepsilon K^A$ (which follow easily from the definition of W_ε), and also (2.4). The upshot of this is that the left-hand sides of the relations (3.5) define objects in $R^\varepsilon{}_{1,3}$ whereas the right-hand sides sit inside $\ell_6{}^\varepsilon$; in fact since the quantities $\{\gamma^i\}$ are more primitive than the $\{J^A, K^A\}$, the relations (3.5) define $[\ ,\ ]_\varepsilon$ via $\underset{\varepsilon}{*}$. The limit $R^\varepsilon{}_{1,3} \to R_{1,0,3}$ then clearly induces the limit $\ell_6{}^\varepsilon \to g_6$.

Further details and comments appear in section 4.

III.4. NOTES

§1.

The discussion given in this section is so much a part of the folk-lore of relativistic physics that it is difficult to assign credit to the various ideas. Indeed one is likely to see some justification of the statement " $L_6^c \to G_6$ as $c \to \infty$ " as soon as one meets the general Lorentz transformation (1.4) (taking $c \to \infty$ in (1.4), the Galileian transformation law immediately pops out).

Not surprisingly, all this was well known to the founders of relativity and in particular, in reasonably sophisticated terms, to Minkowski (1908). On the limit $L_6^c \to G_6$, his epistemological remarks concerning the role of mathematics in the development of our perceptions of nature are poignant to say the least; this ought to be required reading of every mathematician! In the same vein, the article by Dyson (1972) addressing the points made by Minkowski (as well as broader issues) is particularly enjoyable.

§2.

Proposition III.2.1 may be extended in an obvious way. Suppose $V = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ and $W_\varepsilon = \mathrm{id}_{V_0} \oplus \varepsilon \cdot \mathrm{id}_{V_1} \oplus \cdots \oplus \varepsilon^k \cdot \mathrm{id}_{V_k}$, then $\lim_{\varepsilon \to 0} \mathcal{L}^\varepsilon$ exists if and only if $[V_i, V_j] \subset V_0 \oplus V_1 \oplus \cdots \oplus V_{i+j}$ for all i,j such that $i+j < k$. (Mimic the proof of Prop. III.2.1 with the obvious alterations).

When W_ε is of the above form, $\lim_{\varepsilon \to 0} \ell^\varepsilon$ (if it exists) is called a *generalized contraction.*

As an example of this scheme, consider the de Sitter Lie algebra $\mathcal{s}_{10}$ with basis $\{J^A, K^A, P^A, H\}$ and Lie relations (4.13) (Ch. I). Defining, $V_0 = \mathrm{span}_{\mathbb{R}}\{J^A\}$, $V_1 = \mathrm{span}_{\mathbb{R}}\{K^A, P^A\}$ and $V_2 = \mathrm{span}_{\mathbb{R}}\{H\}$, the relations $[V_i, V_j] \subset V_0 \oplus V_1 \oplus \cdots \oplus V_{i+j}$ are readily verified and consequently $\lim_{\varepsilon \to 0} \mathcal{s}_{10}^{\varepsilon}$ exists and in fact equals the Lie algebra of the Carroll group (see Ch. I, (5.6) with a notational change $\Theta \to -H$ and comments following); the easy explicit calculation may be found in Brooke (1978, §6) as well as a slightly modified contraction process $\ell_6 \to g_6$ with the property that $\lim_{\varepsilon \to 0} W_\varepsilon(J^A_L) = J^A_G$ and $\lim_{\varepsilon \to 0} W_\varepsilon(K^A_L) = K^A_G$.

For further interesting discussions see Hermann (1966), Bacry and Lévy-Leblond (1968) and Gilmore (1974).

<u>§3.</u>

For a slightly different treatment of Cor. III.3.3., in which $\{\lim_{\varepsilon \to 0} W_\varepsilon(\gamma^\alpha) : 0 \leq \alpha \leq 3\}$ form an orthonormal basis of $\mathbb{R}^{1,0,3}$ generating $\mathbb{R}_{1,0,3}$, see Brooke (1978, §6). In this treatment, the contraction $\mathbb{R}_{1,3} \to \mathbb{R}_{1,0,3}$ again induces the contraction $\ell_6 \to g_6$ and additionally $\lim_{\varepsilon \to 0} W_\varepsilon(J^A_L) = J^A_G$ and $\lim_{\varepsilon \to 0} W_\varepsilon(K^A_L) = K^A_G$ (see remarks of §2, section 4).

In analogy with the Lie algebra contraction of the de Sitter Lie algebra to the Carroll Lie algebra as described in §2 (this section), there is a Clifford algebra contraction $\mathbb{R}_{1,4} \to \mathbb{R}_{2,0,3}$ inducing this

Lie contraction. Using Thm. III.3.2, define $W_\varepsilon(\gamma^0) = \varepsilon \cdot \gamma^0$, $W_\varepsilon(\gamma^4) = \varepsilon \cdot \gamma^4$, $W_\varepsilon(\gamma^A) = \gamma^A$ where $\{\gamma^0, \gamma^4, \gamma^A : 1 \leq A \leq 3\}$ is an orthonormal basis of $\mathsf{R}^{1,4}$ generating $\mathsf{R}_{1,4}$ ($(\gamma^0)^2 = 1$, $(\gamma^A)^2 = (\gamma^4)^2 = -1$ etc.) and verify that $\lim\limits_{\varepsilon \to 0} \mathsf{R}_{1,4} = \mathsf{R}_{2,0,3}$.

Concerning the notion of Clifford algebra contraction in general, many questions remain unanswered. What can be said about general contractions when W_ε has a more complicated form than that of Thm. III. 3.2.? Fixing one form of W_ε's and given a Clifford algebra C, which Clifford algebras contract to C? (Might one have results analogous to those of Bacry and Lévy-Leblond (1968) in the case of Lie algebras?). What are the deeper connections between the Lie and Clifford contractions? Is every Lie contraction induced by a Clifford contraction? Specifically, if there a "Clifford-like" contraction which induces the Lie contraction of ℓ_{11} to $g_{11}(m)$? (See Ch. I, section 6; also Saletan (1961) for the contraction $\ell_{11} \to g_{11}(m)$.) Finally, and most importantly, is the idea of Clifford algebra contraction interesting and/or useful?

CHAPTER IV

PHYSICAL APPLICATIONS: WAVE EQUATIONS

IV.0. INTRODUCTION

Whereas the quantum mechanical equation describing a relativistic electron (a particle with spin) has been known almost since the birth of wave mechanics (Dirac (1928)), the corresponding non-relativistic equation was only understood on the same fundamental level in comparatively recent times (Lévy-Leblond (1967)). Prior to this, the non-relativistic electron equation was obtained from the relativistic Dirac equation by ad hoc and poorly justified methods involving the taking of the limit c (= speed of light) $\to \infty$; this had the unfortunate effect of suggesting that the quantum mechanical concept of spin is an intrinsically relativistic notion and that the correct value of the magnetic moment of the electron is determined only by relativistic theory. (This latter issue has received very recent attention, from quite a different point of view than Lévy-Leblond's, in the paper of Niederer and O'Raifeartaigh (1977); their conclusions also differ somewhat from his.) It is interesting to note, however, that Pauli (1927) came up with an equation for the non-relativistic spinning electron employing the $SU(2) \cong Spin(3)$ representation of the rotation part of the homogeneous Galilei group $R^3 \circledS SO(3)$; this equation, which predates the Dirac equation, is a second-order matrix partial differential equation generalizing the Schrödinger equation as opposed to the Dirac equation which is of first order. With the perfect clarity of hindsight,

it is in a way surprising that neither Pauli nor Dirac apparently obtained, in a manner analogous to Dirac's derivation of the relativistic equation, a first-order matrix partial differential equation describing the non-relativistic situation. Had they done so, confusion concerning the non-relativistic status of spin would not likely have arisen.

In section 1 of this chapter we recall the work of Lévy-Leblond on wave equations for non-relativistic particles of spin 1/2 and offer various remarks of a critical nature concerning them. As an application of the results of earlier chapters, we propose, in section 2, an alternative equation which possesses improved invariance properties and which, through the use of the de Sitter spin group, takes on an appearance closer to that of the Dirac equation (suitably recast in a form which exploits the relationships I.(3.1) and I.(3.3) between the Lorentz and de Sitter groups). This form of the Dirac equation is displayed in section 3 which also contains a few remarks on the previous material.

IV.1. LÉVY-LEBLOND'S EQUATION

Following the ideas used by Dirac in deriving his relativistic equation for the electron, Lévy-Leblond (1967) has obtained a non-relativistic analogue from which he argues the non-relativistic nature of the magnetic moment of the electron.

We describe his development.

Corresponding to the classical internal energy $E - \frac{1}{2m}\vec{p}\cdot\vec{p}$ of a non-relativistic particle, one defines the *Schrödinger operator* S by the usual replacements: $E = i\partial_t$, $\vec{p} = -i\nabla$; that is, $S = i\partial_t + \frac{1}{2m}\Delta$ where $\Delta = \nabla\cdot\nabla$ is the Laplacian operator. Since m is a Galileian-invariant quantity and therefore transforms as a constant in any representation, it is convenient to work with $2mS$ instead of S (c.f. Prop. III. 1.1). The idea is to find first-order operators $(E = i\partial_t$, $\vec{p} = -i\nabla$):

$$\theta = AE + \vec{B}\cdot\vec{p} + C \tag{1.1a}$$

$$\theta' = A'E + \vec{B}'\cdot\vec{p} + C' \tag{1.1b}$$

with matrix coefficients $A,A',\vec{B},\vec{B}',C,C'$ and which satisfy the operator equality:

$$\theta'\theta = 2mS \tag{1.2}$$

For then, if Φ satisfies the equation:

$$\theta\Phi = 0 \tag{1.3}$$

it will also satisfy the Schrödinger equation:

$$S\Phi = 0 \tag{1.4}$$

The equation $\theta\Phi = 0$ is, in analogy to the Dirac equation, then proposed to describe a non-relativistic particle with spin.

Now the condition (1.2) is equivalent to:

$$A'A = 0 \qquad A'B^K + B'^K A = 0 \tag{1.5a}$$

$$A'C + C'A = 2m \qquad B'^K B^L + B'^L B^K = -2\delta^{KL} \tag{1.5b}$$

$$B'^K C + C'B^K = 0 \qquad C'C = 0 \tag{1.5c}$$

with $1 \leq K , L \leq 3$, or equivalently:

$$B'^i B^j + B'^j B^i = -2\delta^{ij} \tag{1.6}$$

with $1 \leq i , j \leq 5$ and $B^4 = i(A + \frac{1}{2m} C)$, $B'^4 = i(A' + \frac{1}{2m} C')$ and $B^5 = A - \frac{1}{2m} C$, $B'^5 = A' - \frac{1}{2m} C'$. Defining new matrices γ^α for $1 \leq \alpha \leq 4$ by:

$$B^\alpha = iB^5\gamma^\alpha \quad , \quad B'^\alpha = -i\gamma^\alpha B'^5 \tag{1.7}$$

one then obtains:

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta^{\alpha\beta} \tag{1.8}$$

The relations (1.8) are those defining a Clifford algebra for $R^{4,0}$ all of whose $C(4)$ representations are irreducible and equivalent. Choosing

$$\gamma^4 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \quad \text{and} \quad \gamma^K = i \begin{pmatrix} 0 & -\sigma^K \\ \sigma^K & 0 \end{pmatrix}, \quad 1 \leq K \leq 3, \text{ with } \{\sigma^1, \sigma^2, \sigma^3\}$$

the Pauli matrices (see remarks preceding Thm. II. 2.6 for their definition) and $B^5 = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}$, one has:

$$B^K = \begin{pmatrix} \sigma^K & 0 \\ 0 & \sigma^K \end{pmatrix}, \quad B^4 = \begin{pmatrix} 0 & i1_2 \\ i1_2 & 0 \end{pmatrix} \tag{1.9}$$

so that:

$$A = \begin{pmatrix} 0 & 0 \\ 1_2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2m1_2 \\ 0 & 0 \end{pmatrix} \tag{1.10}$$

Finally with $\Phi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, where ϕ, χ each have two components, the equation (1.3) becomes:

$$E\phi + (\vec{\sigma}\cdot\vec{p})\chi = 0 \tag{1.11a}$$

$$(\vec{\sigma}\cdot\vec{p})\phi + 2m\,\chi = 0 \tag{1.11b}$$

again with $E = i\partial_t$, $\vec{p} = -i\nabla$.

Concerning Galilei invariance of (1.11), corresponding to $(b,\vec{a},\vec{v},R) \in G_{10}$ one defines

$$(\vec{x}', t') = (R\vec{x}+\vec{v}t+\vec{a}, t+b) \quad , \quad f(\vec{x}, t) = \frac{1}{2} m|\vec{v}|^2 t + m\vec{v}\cdot R\vec{x}$$

and:

$$\Phi'(\vec{x}',t') = e^{if(\vec{x},t)}\,\Delta^{1/2}(\vec{v},R)\Phi(\vec{x},t) \tag{1.12}$$

where $\Phi' = \begin{pmatrix}\phi' \\ \chi'\end{pmatrix}$, $\Delta^{1/2}(\vec{v},R) = \begin{pmatrix} D^{1/2}(R) & 0 \\ -\frac{\vec{\sigma}\cdot\vec{v}}{2}D^{1/2}(R) & D^{1/2}(R)\end{pmatrix}$ and $D^{1/2}$ is the SU(2) representation of SO(3) defined by $D^{1/2}(R^{-1})\sigma^A D^{1/2}(R) = R^A{}_B\sigma^B$ ($\Delta^{1/2}$ has already been encountered in Ch. II, §4). The mapping $\Phi \to \Phi'$ given by (1.12) defines a representation of G_{10} which leaves (1.11) invariant:

$$\begin{aligned} &i\partial_t\phi + \frac{1}{i}\vec{\sigma}\cdot\nabla\chi = 0 && && i\partial_{t'}\phi' + \frac{1}{i}\vec{\sigma}\cdot\nabla'\chi' = 0 \\ & && \Longleftrightarrow && \\ &\frac{1}{i}\vec{\sigma}\cdot\nabla\phi + 2m\chi = 0 && && \frac{1}{i}\vec{\sigma}\cdot\nabla'\phi' + 2m\chi' = 0 \end{aligned} \tag{1.13}$$

Because of this invariance property, $\theta\Phi = 0$ is claimed to adequately describe a free, non-relativistic particle with mass m and spin $1/2$.

A few comments are in order: first of all concerning the relations (1.6). While indeed it is true that all $C(4)$ representations of the relations (1.8) are equivalent, the same is not true for (1.6) and consequently, contrary to Lévy-Leblond's claim (Lévy-Leblond (1971), p. 286), not all choices for $A,\vec{B},C$ are equivalent (i.e. related by a similarity. In fact, if $\{\gamma^\alpha\}$ satisfy (1.8) and B^5 is any non-singular matrix, define $B^\alpha = iB^5\gamma^\alpha$ and $B'^i = -(B^i)^{-1}$ to arrive at a representation of (1.6). Thus (1.6) is determined by B^i , $1 \leq i \leq 5$, or just the same by γ^α and B^5 , $1 \leq \alpha \leq 4$, with $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta^{\alpha\beta}$ and B^5 non-singular but otherwise arbitrary. It is now quite easy to see that not all choices of γ^α and B^5 are equivalent). Unfortunately this claim appears elsewhere in the literature (see Hagen (1970), p. 98). A

consequence of this inequivalence will be discussed shortly but before that, a comparison between the relativistic and non-relativistic wave equations for spin 1/2 emphasizing common features will be discussed.

Lorentz

We consider operators $\theta = EA + c\vec{p}\cdot\vec{B} + mc^2C$ and $\theta' = EA' + c\vec{p}\cdot\vec{B}' + mc^2C'$ such that:

$$\theta'\theta = E^2 - c^2\vec{p}\cdot\vec{p} - m^2c^4 \tag{1.14}$$

(the right-hand side of (1.14) is the free Klein-Gordon operator). The condition (1.14) is equivalent to:

$$B'^0 = (B^0)^{-1} \;,\; B'^K = -(B^K)^{-1} \;,\; B'^4 = -(B^4)^{-1} \tag{1.15a}$$

$$(B^K)^{-1}B^L + (B^L)^{-1}B^K = 2\delta^{KL} \quad,\quad (B^4)^{-1}B^K + (B^K)^{-1}B^4 = 0 \tag{1.15b}$$

$$(B^0)^{-1}B^K - (B^K)^{-1}B^0 = 0 \quad,\quad (B^0)^{-1}B^4 - (B^4)^{-1}B^0 = 0 \tag{1.15c}$$

where $B^0 = A$, $B^4 = C$ and $B'^0 = A'$, $B'^4 = C'$ and $1 \leq K , L \leq 3$.

Galilei

Here we want to find $\theta = EA + \vec{p}\cdot\vec{B} + mC$ and $\theta' = EA' + \vec{p}\cdot\vec{B}' + mC'$ such that:

$$\theta'\theta = 2mE - \vec{p}\cdot\vec{p} \tag{1.16}$$

Defining $B^0 = \frac{1}{\sqrt{2}}(A+C)$, $B^4 = \frac{1}{\sqrt{2}}(A-C)$ and $B'^0 = \frac{1}{\sqrt{2}}(A'+C')$, $B'^4 = \frac{1}{\sqrt{2}}(A'-C')$ the relations:

$$B'^0 = (B^0)^{-1} \quad , \quad B'^K = -(B^K)^{-1} \quad , \quad B'^4 = -(B^4)^{-1} \tag{1.17a}$$

$$(B^K)^{-1}B^L + (B^L)^{-1}B^K = 2\delta^{KL} \quad , \quad (B^4)^{-1}B^K + (B^K)^{-1}B^4 = 0 \tag{1.17b}$$

$$(B^0)^{-1}B^K - (B^K)^{-1}B^0 = 0 \quad , \quad (B^0)^{-1}B^4 - (B^4)^{-1}B^0 = 0 \tag{1.17c}$$

with $1 \leq K , L \leq 3$, are equivalent to (1.16).

Comparing the Lorentz and Galilei situations we see that relations (1.15) and (1.17) are identical!; both may be written:

$$B'^iB^j + B'^jB^i = -2g^{ij} \tag{1.18}$$

where $(g^{ij}) = \mathrm{diag}(-1,1,1,1,1)$ is the de Sitter "metric". A particularly promising choice of $\{B'^i, B^i\}$ is $B'^i = B^i$ satisfying the conditions $B^iB^j + B^jB^i = -2g^{ij}$ defining a Clifford algebra for $\mathbb{R}^{1,4}$; with this choice by the way, $\theta' = \theta$ and so θ^2 is either the free Klein-Gordon or free Schrödinger operator for a particle of mass m . Furthermore with this choice $(\theta' = \theta)$, we know the B^i 's up to equivalence by Prop. II.4.2; still further, with this choice there is a more satisfactory implementation of the appropriate kinematical group as will now be explained and this brings us to the second comment concerning Lévy-Leblond's equation.

While it is true that Lévy-Leblond's equation is invariant (see (1.13)), his operator θ is not. Generally, given a group represen-

tation on a set of functions on which a differential operator acts, the operator is said to be *invariant* if it commutes with the group representation (see Hermann (1966), p. 78). To be specific, Lévy-Leblond's operator θ_x is given by:

$$\theta_x = \begin{pmatrix} \frac{1}{i}\vec{\sigma}\cdot\nabla_{\vec{x}} & 2m \\ i\partial_t & \frac{1}{i}\vec{\sigma}\cdot\nabla_{\vec{x}} \end{pmatrix} \tag{1.19}$$

where $x = (\vec{x},t)$. If θ were to be invariant, then it would have to happen that for $(\vec{v},R) \in G_6$, $x = (\vec{x},t)$ and $x' = (\vec{x}',t') = (R\vec{x}+\vec{v}t,t)$:

$$\theta_{x'} \circ e^{if(x)}\Delta^{1/2}(\vec{v},R) = e^{if(x)}\Delta^{1/2}(\vec{v},R) \circ \theta_x \tag{1.20}$$

However (1.20) is not satisfied but instead one has:

$$\theta_{x'} \circ e^{if(x)}\Delta^{1/2}(\vec{v},R) = \Delta^{1/2}(-2\vec{v},1_3) \circ e^{if(x)}\Delta^{1/2}(\vec{v},R) \circ \theta_x \tag{1.21}$$

(This, incidentally, proves the invariance of the equation i.e. $\theta_{x'}\Phi'(\vec{x}',t') = 0 \Leftrightarrow \theta_x\Phi(\vec{x},t) = 0$). The operator $\begin{pmatrix} i\partial_t & \frac{1}{i}\vec{\sigma}\cdot\nabla_{\vec{x}} \\ \frac{1}{i}\vec{\sigma}\cdot\nabla_{\vec{x}} & 2m \end{pmatrix}$ is also used by Lévy-Leblond (1971) and Hagen (1970); neither is it invariant under the representation (1.12), contrary to a statement of Hagen (1970, p. 101). As long as one considers only free-particle equations (and perhaps also those corresponding to particles with minimal electromagnetic coupling), one is not likely to encounter difficulties by having a non-invariant operator (even though, for example, the Klein-Gordon and Schrödinger operators are invariant). However it seems less clear that such difficulties will not arise (at least in interpretation) if more complicated interactions are introduced.

The point to be made is this. If one demands of θ that it satisfy $\theta^2 = 2mE - \vec{p}\cdot\vec{p}$ instead of the Lévy-Leblond condition, then with respect to the obvious representation of the Galilei spin group Spin(1,0,3) on wave functions, the operator θ is invariant, a similar statement holds in the relativistic case. Details follow in the next section.

IV.2. THE NON-RELATIVISTIC, SPIN 1/2 WAVE EQUATION

As may be inferred from previous comments (Ch. I, Notes for §3 and Ch. III, section 1), the homogeneous Galilei group G_6 is isomorphic to a set of linear transformations of $\mathbb{R}^5$ leaving **fixed** the vector $\begin{pmatrix} 0 \\ \vec{0} \\ 1 \end{pmatrix}$ and the bilinear form $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1_3 & 0 \\ -1 & 0 & 0 \end{pmatrix}$; in fact

$$G_6 \cong \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \vec{v} & R & 0 \\ \frac{1}{2}|\vec{v}|^2 & \vec{v}^{\,t}R & 1 \end{pmatrix} : \vec{v} \in \mathbb{R}^3 \, , \, R \in SO(3) \right\}$$

whose action on $\mathbb{R}^5 = \left\{ \begin{pmatrix} m \\ \vec{p} \\ E \end{pmatrix} \right\}$ is defined by:

$$\hat{m} = m \tag{2.1a}$$

$$\hat{\vec{p}} = R\vec{p} + m\vec{v} \tag{2.1b}$$

$$\hat{E} = E + \vec{v}\cdot R\vec{p} + \frac{1}{2}\, m|\vec{v}|^2 \tag{2.1c}$$

If by $B = (B^{ab})$, $0 \le a$, $b \le 4$, we mean the (non-degenerate) bilinear form $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1_3 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, then define p_a by $p^a = B^{ab}p_b$ (explicitly, $p_0 = -p^4 = -E$, $p_A = p^A$, $p_4 = -p^0 = -m$) so that $p_a p^a = -2mE + \vec{p}\cdot\vec{p}$. Moreover, $\Lambda \in G_6$ acts on $\mathbb{R}^5 = \{(-E,\underset{\to}{p},-m)\}$ (where as usual $\underset{\to}{p} = (p_1,p_2,p_3)$, as opposed to $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}$) by

$$p_a \to \hat{p}_a = p_b(\Lambda^{-1})^b{}_a \tag{2.2}$$

Let γ^{α} , $0 \leq \alpha \leq 4$, be an orthonormal basis of (R^5, B) generating a (possibly non-universal) Clifford algebra for (R^5, B); then γ^{α} , $0 \leq \alpha \leq 3$, form an orthonormal basis of $R^{1,0,3}$ and generate a Clifford algebra for $R^{1,0,3}$. The embedding $\{\gamma^{\alpha} : 0 \leq \alpha \leq 3\} \subset \{\gamma^{\alpha} : 0 \leq \alpha \leq 4\}$ induces the embedding $\mathrm{Spin}(1,0,3) \subset \mathrm{Spin}^+(B) \cong \mathrm{Spin}^+(1,4)$ which is the spin group analogue of $G_6 \subset SO^+(B) \cong SO^+(1,4)$ as expressed as in (2.1) or (2.2). It is the spin group analogue of the action of G_6 on R^5 by (2.1) that we must ascertain, and while it is possible to do so by employing abstract arguments only, it is convenient to use an explicit matrix representation. In an $H(2)$ representation we choose:

$$\gamma^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{2.3a}$$

$$\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.3b}$$

$$\gamma^2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \tag{2.3c}$$

$$\gamma^3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \tag{2.3d}$$

$$\gamma^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \tag{2.3e}$$

(which may be obtained from (2.1) of Ch. II by the replacements $\frac{1}{\sqrt{2}}(\gamma^0+\gamma^4)$, γ^A , $\frac{1}{\sqrt{2}}(\gamma^0-\gamma^4)$ for $\gamma^0, \gamma^A, \gamma^4$ respectively). Then, as in Thm. II.2.7 (with β replaced by $\frac{1}{\sqrt{2}}\beta$):

$$\mathrm{Spin}(1,0,3) = \left\{ \begin{pmatrix} \alpha - \frac{1}{\sqrt{2}}\beta & -\frac{1}{\sqrt{2}}\beta \\ \frac{1}{\sqrt{2}}\beta & \alpha + \frac{1}{\sqrt{2}}\beta \end{pmatrix} : |\alpha| = 1 \ , \ \bar{\alpha}\beta + \bar{\beta}\alpha = 0 \right\} .$$

We know from Ch. I (3.7), (3.9) that if $g = \begin{pmatrix} \alpha - \frac{1}{\sqrt{2}}\beta & -\frac{1}{\sqrt{2}}\beta \\ \frac{1}{\sqrt{2}}\beta & \alpha + \frac{1}{\sqrt{2}}\beta \end{pmatrix}$

induces $\Lambda \in G_6$ (as defined by (2.1)) according to $g^{-1}\gamma^a g = \Lambda^a{}_b\gamma^b$, then $R = ((a^0)^2 - |\vec{a}|^2)1_3 + 2|\vec{a}|^2 \ P_{\vec{a}} + 2a^0 \ J_{\vec{a}}$, $\vec{v} = 2(-a^0\vec{b} + b^0\vec{a} - \vec{a}\times\vec{b})$ where $\alpha = a^0 + ia^1 + ja^2 + ka^3$ and $\beta = b^0 + ib^1 + jb^2 + kb^3$. Thus in addition to:

$$\hat{\gamma}^0 = g^{-1}\gamma^0 g = \gamma^0 \tag{2.4a}$$

$$\hat{\gamma}^A = g^{-1}\gamma^A g = v^A\gamma^0 + R^A{}_B\gamma^B \tag{2.4b}$$

we then have (by a somewhat lengthy calculation):

$$\hat{\gamma}^4 = g^{-1}\gamma^4 g = \gamma^4 + v^A\delta_{AB}R^B{}_C\gamma^C + \frac{1}{2}|\vec{v}|^2\gamma^0 \tag{2.4c}$$

and as a result, $\hat{\gamma}^a\hat{p}_a = \gamma^a p_a$ is an invariant quantity.

Before proceeding further, to avoid potential confusion that might arise in the ordering of factors involving members of H and the operator forms of E and $\vec{p}$, we shall adopt a $\mathrm{C}(4)$ representation (for example: $\gamma^0 = \frac{1}{\sqrt{2}}(\tilde{\gamma}^0 + \tilde{\gamma}^4)$, $\gamma^A = \tilde{\gamma}^A$, $\gamma^4 = \frac{1}{\sqrt{2}}(\tilde{\gamma}^0 - \tilde{\gamma}^4)$ with $\tilde{\gamma}^0, \tilde{\gamma}^A, \tilde{\gamma}^4$ as in Prop. II.4.2).

Define a representation π of $\mathrm{Spin}(1,0,3)$ on $\mathrm{C}(4)$ - valued functions ψ over space-time by:

$$(\pi(g)\psi)(\Lambda(g)\cdot x) = e^{i\xi_m(g,x)} g\cdot\psi(x) \tag{2.5}$$

where $\Lambda(g)$ is the image in G_6 of $g \in \text{Spin}(1,0,3)$ corresponding to $(\vec{v},R)$; $x = (t,\vec{x})$; and

$$\xi_m(g,x) = m(\frac{1}{2}|\vec{v}|^2 t + \vec{v}\cdot R\vec{x}) \tag{2.6}$$

In place of Lévy-Leblond's operator θ , we propose instead:

$$\underset{G}{D} = \gamma^a p_a{}^{op} \tag{2.7}$$

where $p_\alpha{}^{op} = \frac{1}{i}\partial_{x^\alpha}$, $p_4{}^{op} = -m$ and $x^0 = t$; (this is equivalent to setting $E = i\partial_t$, $\vec{p} = -i\nabla_{\vec{x}}$). The first nice feature about $\underset{G}{D}$ is that, by construction, it is invariantly defined (recall that $\hat{\gamma}^a\hat{p}_a = \gamma^a p_a$). The second nice feature is the content of

<u>Proposition IV.2.1.</u>: The differential operator $\underset{G}{D}$ is invariant under $\text{Spin}(G_6)$:

$$\underset{G}{D} \circ \pi(g) = \pi(g) \circ \underset{G}{D} \tag{2.8}$$

for all $g \in \text{Spin}(G_6)$.

<u>Proof</u>: Equation (2.8) is equivalent to:

$$\underset{G}{D}{}_{\Lambda x}(e^{i\xi_m(g,x)} g\cdot\psi(x)) = e^{i\xi_m(g,x)} g\cdot \underset{G}{D}{}_x\psi(x) \tag{2.9}$$

for all ψ and x . Writing $\hat{x} = \Lambda\cdot x$ or $\hat{x}^\alpha = \Lambda^\alpha{}_\beta x^\beta$, we have $\partial_{\hat{x}^\alpha} = \partial_{x^\beta}(\Lambda^{-1})^\beta{}_\alpha$ and consequently $\gamma^\alpha\partial_{\hat{x}^\alpha} = \gamma^\alpha\partial_{x^\beta}(\Lambda^{-1})^\beta{}_\alpha = (\Lambda^{-1})^\beta{}_\alpha\gamma^\alpha\partial_{x^\beta} = g\gamma^\beta g^{-1}\partial_{x^\beta}$ (recalling (2.4 a,b) with g^{-1} replacing g). As a result:

$$\underset{G}{D}{}_{\Lambda\cdot x} = \underset{G}{D}{}_{\hat{x}} = \frac{\gamma^\alpha}{i}\,\partial_{\hat{x}^\alpha} - m\gamma^4$$

$$= g\left(\frac{\gamma^\alpha}{i}\,\partial_{x^\alpha}\right)g^{-1} - m\gamma^4$$

$$= g\,\underset{G}{D}{}_x g^{-1} + mg(\gamma^4 - \hat{\gamma}^4)g^{-1} \quad .$$

Evaluating the left side of (2.9):

$$\underset{G}{D}{}_{\Lambda\cdot x}\left(e^{i\xi_m(g,x)}\,g\cdot\psi(x)\right) = g\left(\underset{G}{D}{}_x\left(e^{i\xi_m(g,x)}\,\psi(x)\right) + m(\gamma^4 - \hat{\gamma}^4)e^{i\xi_m(g,x)}\,\psi(x)\right)$$

$$= e^{i\xi_m(g,x)}\,g\cdot\underset{G}{D}{}_x\psi(x) \quad +$$

$$g\left(\frac{\gamma^\alpha}{i}\,\partial_{x^\alpha}\left(e^{i\xi_m(g,x)}\right) + m(\gamma^4 - \hat{\gamma}^4)e^{i\xi_m(g,x)}\right)\psi(x)$$

but

$$\frac{\gamma^\alpha}{i}\,\partial_{x^\alpha}\left(e^{i\xi_m(g,x)}\right) = e^{i\xi_m(g,x)}\left(\frac{1}{2}\,m|\vec{v}|^2\gamma^0 + m\vec{v}\cdot R\vec{\gamma}\right)$$

$$= e^{i\xi_m(g,x)}\,m(\hat{\gamma}^4 - \gamma^4)$$

by (2.4c) and therefore

$$g\left(\frac{\gamma^\alpha}{i}\,\partial_{x^\alpha}\left(e^{i\xi_m(g,x)}\right) + m(\gamma^4 - \hat{\gamma}^4)e^{i\xi_m(g,x)}\right)\psi(x) = 0$$

implying the validity of (2.9) and hence of (2.8). □

In order to compare $\underset{G}{D}$ with Lévy-Leblond's θ, consider the representation of $\mathrm{Spin}(G_6)$ according to Cor. II.4.6 with b replaced by $\frac{1}{\sqrt{2}}\,b$; conjugate this representation by $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1_2 & 1_2 \\ -1_2 & 1_2 \end{pmatrix}$, and the

γ^a 's by U also to obtain:

$$\mathrm{Spin}(G_6) = \left\{ \begin{pmatrix} a & 0 \\ \sqrt{2}\, b & a \end{pmatrix} : a,b \in \chi(H) \ , \ a^* a = 1_2 \ , \ a^* b + b^* a = 0 \right\}$$

and

$$U\gamma^0 U^{-1} = \begin{pmatrix} 0 & 0 \\ -\sqrt{2}\, 1_2 & 0 \end{pmatrix} , \quad U\gamma^A U^{-1} = \begin{pmatrix} \tau^A & 0 \\ 0 & -\tau^A \end{pmatrix} , \quad U\gamma^4 U^{-1} = \begin{pmatrix} 0 & -\sqrt{2}\, 1_2 \\ 0 & 0 \end{pmatrix}$$

Then

$$U \underset{Gx}{\mathrm{D}} U^{-1} = \begin{pmatrix} -\vec{\sigma}\cdot\nabla_{\vec{x}} & \sqrt{2}\, m \\ \sqrt{2}\, i\partial_t & \vec{\sigma}\cdot\nabla_{\vec{x}} \end{pmatrix}$$

so that with $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the wave equation $\underset{G}{\mathrm{D}}\psi = 0$ becomes:

$$(\vec{\sigma}\cdot\nabla_{\vec{x}})\psi_1 - \sqrt{2}\, m\, \psi_2 = 0$$

$$\sqrt{2}\, i\partial_t\, \psi_1 + (\vec{\sigma}\cdot\nabla_{\vec{x}})\psi_2 = 0$$

(which together imply the Schrödinger equations:

$$i\partial_t\, \psi_1 = -\frac{1}{2m}\Delta_{\vec{x}}\, \psi_1 \quad \text{and} \quad i\partial_t\, \psi_2 = -\frac{1}{2m}\Delta_{\vec{x}}\, \psi_2 \quad).$$

A few words are in order concerning $U \underset{Gx}{\mathrm{D}} U^{-1}$.

First of all, by Prop. II.4.2., the representations:

$$\gamma^0 = \frac{\kappa}{2}(\tilde{\gamma}^0 + \tilde{\gamma}^4) \ , \quad \gamma^A = \tilde{\gamma}^A \ , \quad \gamma^4 = \frac{\kappa}{2}(\tilde{\gamma}^0 - \tilde{\gamma}^4) \ , \quad \text{for } \kappa \neq 0 \ ,$$

are equivalent. Making such a choice has the effect of replacing the $\sqrt{2}$ by κ in the above representation of $\mathrm{Spin}(G_6)$ and also in the above matrix representation of $U \underset{Gx}{\mathrm{D}} U^{-1}$.

Secondly, to reiterate, by Prop. IV.2.1. $U \underset{G}{D} U^{-1}$ is invariant under the representation π of Spin(G_6) given by (2.5) (where now Spin(G_6) refers to the equivalent representation obtained by replacing the $\sqrt{2}$ by κ as just outlined above). The operator $U \underset{G}{D} U^{-1}$ is invariant because the representation of the γ^a 's and the representation of Spin(G_6) are compatible (indeed $U \underset{G}{D} U^{-1}$ is constructed directly from the γ^a 's). This is in contrast with Lévy-Leblond's operator θ (1.19) which does not bear any obvious relation to his $\Delta^{1/2}$ representation. Consequently, it is not surprising that his θ is not invariant in the sense of (1.20); invariance of an operator depends crucially on the given representation.

Supposing now that the particle defined by ψ possesses an electric charge e, the electromagnetic interaction is introduced as usual by minimal coupling: $p_\alpha \to p_\alpha - e\,A_\alpha$ (with A_0 the electric potential and $\underset{\to}{A}$ the magnetic potential). Denoting by ∇_α the first-order operator $\partial_{x^\alpha} - ie\,A_\alpha$, we write $\underset{Gx}{\tilde{D}} = \frac{\gamma^\alpha}{i} \nabla_\alpha - m\gamma^4$. The equation $\underset{G}{\tilde{D}}\psi = 0$ is then supposed to describe a charged, non-relativistic particle of spin 1/2 in an electromagnetic field.

The Pauli equation results from $\underset{G}{\tilde{D}}^2\psi = 0$. In fact, defining $F_{\alpha\beta} = \partial_{x^\alpha} A_\beta - \partial_{x^\beta} A_\alpha$ and $S^{\alpha\beta} = \frac{1}{2}[\gamma^\alpha,\gamma^\beta]$ (the electromagnetic field and spin tensors respectively):

$$\underset{G}{\tilde{D}}^2 = -\gamma^\alpha\gamma^\beta\nabla_\alpha\nabla_\beta + im(\gamma^\alpha\gamma^4+\gamma^4\gamma^\alpha)\nabla_\alpha + m^2(\gamma^4)^2$$

$$= \gamma^{\alpha\beta}\nabla_\alpha\nabla_\beta + i\ 2m\ \nabla_0 - \frac{1}{2}\ ieS^{\alpha\beta}F_{\alpha\beta}$$

$$= \nabla^\alpha\nabla_\alpha + i\ 2m\ \nabla_0 - \frac{1}{2}\ ieS^{\alpha\beta}F_{\alpha\beta}$$

where $\nabla^{\alpha} = \gamma^{\alpha\beta}\nabla_{\beta}$ ($(\gamma^{\alpha\beta}) = \mathrm{diag}(0,1,1,1)$), and use has been made of the identities: $\gamma^{\alpha}\gamma^{\beta} = -\gamma^{\alpha\beta} + S^{\alpha\beta}$, $[\nabla_{\alpha},\nabla_{\beta}]\psi = ieF_{\alpha\beta}\psi$. See Messiah (1966, Ch. XX §21) and Berestetskiĭ, Lifshitz, Pitaevskiĭ (1971, Ch. IV) for further details.

IV.3. NOTES

§1.

Although Lévy-Leblond's operator θ is not invariant, the existence of distinct invariant operators θ', θ satisfying $\theta'\theta = 2mS$ has not been ruled out. To understand what is involved one would have to examine the relations (1.17) more closely; this is yet to be done.

The general problem of factorizing differential operators has not received a great deal of mathematical attention, although in the relativistic context some work has been done; operators θ, θ' which factorize the Klein-Gordon operator $-\partial_t^{\,2} + \Delta - m^2$ are called Klein-Gordon divisors (see Takahashi, Y. (1969) for a discussion of this and for earlier references). Also motivated by matters relativistic, Hermann (1975, Ch. XX) has developed some results on the factorization of differential operators on pseudo-Riemannian manifolds likely to be of interest when it is desired to formulate quantum mechanics on curved space-times.

§2.

In the relativistic case we proceed along similar lines. Let $D_L = \gamma^a p_a^{\text{op}}$ where now $\gamma^a\gamma^b + \gamma^b\gamma^a = -2g^{ab}$ ($(g^{ab}) = \text{diag}(-1,1,1,1,1)$) and as before $p_\alpha^{\text{op}} = \frac{1}{i}\partial_{x^\alpha}$ but $p_4^{\text{op}} = \varepsilon\, m$ with $\varepsilon = \pm 1$. (The definition of D_L is invariant because under the action of $\text{Spin}^+(L_6) \subset \text{Spin}^+(1,4)$, $\hat{\gamma}^a\hat{p}_a = \gamma^a p_a$, where $p_a g^{ab} = p^b$ and $p^0 = E$, $p^4 = m$). The representation of $\text{Spin}^+(L_6)$ on spinor-valued functions on space-time is given by $(\pi(g)\psi)(\Lambda\cdot x) = g\cdot\psi(x)$, where $\hat{x} = \Lambda\cdot x$, $(\Lambda\cdot x)^\alpha = \Lambda^\alpha{}_\beta x^\beta$

and $g^{-1}\gamma^{a}g = \Lambda^{a}{}_{b}\gamma^{b}$. Invariance of $\underset{L}{D}$ is easy to see:

$$\underset{L}{D}_{\Lambda\cdot x}(\pi(g)\psi)(\Lambda\cdot x) = (\frac{\gamma^{\alpha}}{i}\,\partial_{\hat{x}^{\alpha}} + \varepsilon\cdot m\,\gamma^{4})g\cdot\psi(x)$$

$$= (\frac{\gamma^{\alpha}}{i}\,(\Lambda^{-1})^{\beta}{}_{\alpha}\,\partial_{x^{\beta}} + \varepsilon\cdot m\,\gamma^{4})g\cdot\psi(x)$$

$$= (g\,\frac{\gamma^{\alpha}}{i}\,g^{-1}\,\partial_{x^{\alpha}} + \varepsilon\cdot m\,g\gamma^{4}g^{-1})g\cdot\psi(x)$$

$$= g(\frac{\gamma^{\alpha}}{i}\,\partial_{x^{\alpha}} + \varepsilon\cdot m\,\gamma^{4})\psi(x)$$

$$= (\pi(g)(\underset{L}{D}\psi))(\Lambda\cdot x)$$

where use has been made of the fact that $Spin^{+}(L_{6}) = Spin^{+}(1,4)_{\gamma^{4}}$ i.e. $g\gamma^{4}g^{-1} = \gamma^{4}$.

As a final remark it may be said that the generalization to curved space-times is made considerably easier in an approach using invariant definitions.

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